The Astrophysical Journal, 640:L103–L106, 2006 March 20


# PARALLEL DIFFUSION OF CHARGED PARTICLES IN STRONG TWO-DIMENSIONAL TURBULENCE

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*Received 2005 October 6; accepted 2006 February 1; published 2006 February 28*

## ABSTRACT

Computation of charged-particle orbits shows that large-amplitude two-dimensional magnetic turbulence supports diffusive transport of charged test particles parallel to the mean magnetic field. This stands in sharp contrast to scattering in the quasi-linear approximation, for which we show quite generally that the two-dimensional scattering rate vanishes. We also demonstrate that at large amplitude, two-dimensional turbulence makes important contributions to the parallel mean free path of particles in mixtures of two-dimensional and slab turbulence. This raises important questions regarding cosmic-ray mean free paths that had been thought to be settled based on quasi-linear theory.

*Subject headings:* diffusion — scattering — turbulence

## 1. INTRODUCTION

The classical quasi-linear treatment (QLT) of charged-particle scattering by magnetic turbulence (Jokipii 1966) has remained the cornerstone of our understanding of cosmic-ray transport in the heliosphere and in the Galaxy for almost 40 years. There have been a variety of nonlinear scattering theories (Völk 1973; Matthaeus et al. 2003; Shalchi et al. 2004) that extend QLT, many of them dealing with the difficult theoretical and observational issues surrounding perpendicular transport (Urch 1977; Matthaeus et al. 1995; Kóta & Jokipii 2000). However, QLT, when extended to include subtle effects of more realistic turbulence spectra, has proved to be a highly resilient approach to understanding parallel scattering. For example, it has been suggested (Bieber et al. 1994) that when three new effects are added to standard QLT, it provides a basis for understanding the mean free paths of cosmic rays, inferred from solar energetic particle studies (Bieber et al. 1994; Dröge 2003). These additional effects are (1) steepening of the turbulence spectrum at high wavenumber, (2) dynamical effects, that is, turbulence that decorrelates in time, and (3) spectral anisotropy. The last of these, which is due to the preferred direction imposed by the local mean magnetic field $\boldsymbol{B}_0$ (Shebalin et al. 1983), has led to models, frequently used in heliospheric studies, that consist of a one-dimensional or "slab" contribution that varies only along $\boldsymbol{B}_0$ and a two-dimensional component that varies only in the two directions perpendicular to $\boldsymbol{B}_0$. One feature of these developments has been the assertion, based on QLT, that two-dimensional turbulence makes little or no contribution to the velocity-space diffusion (pitch-angle scattering) that underlies parallel transport. Here we show that this assertion is false for large-amplitude turbulence, even though it is a rigorous consequence of QLT. This suggests that we revisit the theoretical basis of our understanding of observed or inferred mean free paths of solar energetic particles and Galactic cosmic rays.

The diffusion coefficient for pitch-angle scattering of particles moving in a mean magnetic field $\boldsymbol{B}_0$ at speed $V$ and with an (unperturbed) gyrofrequency $\Omega = eB_0/mc$ can be evaluated using the methods introduced by Jokipii (1966). (Here $m = \gamma m_0$, with rest mass $m_0$ and Lorentz factor $\gamma$.) The procedure is to compute the mean square changes in pitch-angle cosine $\mu = V_z/V = \boldsymbol{B}_0 \cdot \boldsymbol{V}/(B_0 V)$ due to perturbations of the magnetic field $\boldsymbol{b}(\boldsymbol{x}, t)$, where the total magnetic field is written as $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}(\boldsymbol{x}, t)$. Note that in magnetostatic scattering the time dependence of $\boldsymbol{b}$ is ignored and electric fields are neglected, so the particle speed $V$ is constant in time. The velocity-space, or pitch-angle, diffusion coefficient can be expressed in the usual form as

$$\phi(\mu) = \lim_{\Delta t \to \infty} \frac{\langle (\Delta\mu)^2 \rangle}{\Delta t}, \qquad (1)$$

where the numerator is the mean square change in pitch angle for an ensemble of particles, which is a function of time. The time increment $\Delta t$, formally expressed as a limit to infinity, is large enough that the pitch-angle displacements are randomized, but not so large that they experience the limitations of the finite $\mu$-space volume. Typically $\Delta t > \lambda_c/V$ is the timescale required (Kaiser et al. 1978), where $\lambda_c$ is the correlation scale. The essence of QLT is to compute the numerator $\langle (\Delta\mu)^2 \rangle$ by iterating once the Newton-Lorentz force law (see § 2). Ignoring the fluctuation in the magnetic field, the solution of $d\boldsymbol{V}/dt = (q/mc)\boldsymbol{V} \times \boldsymbol{B}_0$ leads to the unperturbed helical trajectory $\boldsymbol{X}_Q(t) = (r_L \sin\theta \cos\Omega t + \eta, -r_L \sin\theta \sin\Omega t + \eta, V\mu t) \equiv [\boldsymbol{r}_\perp(t), V\mu t]$ for Larmor radius $r_L = V/\Omega$, with the unperturbed velocity $\boldsymbol{V}_Q = d\boldsymbol{X}_Q/dt$, $\boldsymbol{r}_\perp(t)$ designating the oscillatory transverse displacement, and $\sin\theta = (1-\mu^2)^{1/2}$. One then solves for the perturbation in $\mu$ as

$$\Delta\mu(t) = \frac{V_z(t) - V_z(0)}{V} = \frac{q}{mcV}\int_0^t \hat{\boldsymbol{z}} \cdot [\boldsymbol{V}_Q \times \boldsymbol{b}(\boldsymbol{X}_Q)]\, d\tau, \qquad (2)$$

where $\hat{\boldsymbol{z}}$ is in the $\boldsymbol{B}_0$-direction. In standard QLT one often ignores the variation of the perturbation in the transverse directions $(x, y)$, in which case one finds that the cyclic motion of the particle picks out a resonance at the resonant parallel wavenumber, leading to the standard expressions (Jokipii 1966). Here we will not ignore the transverse variations; instead, let us consider the case in which the magnetic perturbation is strictly two-dimensional, $\boldsymbol{b}(\boldsymbol{x}) = \boldsymbol{b}^{2D}(x, y)$, varying *only* in the transverse directions. The two-dimensional perturbation may be written as

$$\boldsymbol{b}^{2D}(x, y) = \boldsymbol{\nabla} \times \hat{\boldsymbol{z}} a(x, y) = \boldsymbol{\nabla} a(x, y) \times \hat{\boldsymbol{z}}, \qquad (3)$$

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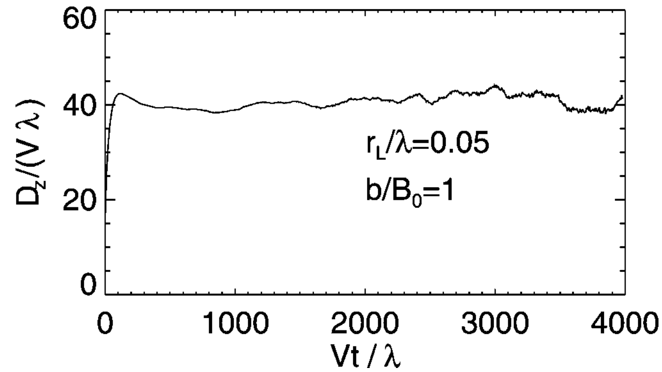


FIG. 1.—Running parallel diffusion coefficient of a test-particle simulation vs. time in pure two-dimensional fluctuations with $r_L/\lambda = 0.05$ and $b/B_0 = 1.0$.

where we introduce the two-dimensional flux function $a(x, y)$ without loss of generality. Note that $\boldsymbol{b}^{2D} \cdot \boldsymbol{B}_0 = 0$. To see how two-dimensional turbulence changes pitch angles in QLT, we evaluate equation (2) using equation (3). For convenience we express the time limit as $t = NT_0 + t'$, a number of gyroperiods $N$ plus a remainder $t' < T_0$, where $T_0 = 2\pi/\Omega$ is the gyroperiod. The limit $t \to \infty$ corresponds to $N \to \infty$. Therefore $\Delta\mu$ consists of two parts,

$$\Delta\mu = \frac{q}{mcV}\left(N\int_0^{T_0} + \int_{NT_0}^{t}\right) d\tau \{\boldsymbol{V}_Q \times [\boldsymbol{\nabla} a(\boldsymbol{X}_Q) \times \hat{\boldsymbol{z}}]\} \cdot \hat{\boldsymbol{z}}, \quad (4)$$

where use is made of the periodicity of $\boldsymbol{V}_Q$ and $\boldsymbol{X}_Q$ in the transverse plane, and the invariance of $a(x, y)$ in the $z$-direction. After applying an elementary vector identity, the first term becomes

$$\frac{qN}{mcV}\hat{\boldsymbol{z}} \cdot \int_0^{T_0} d\tau \frac{d\boldsymbol{X}_Q}{d\tau} \times [\boldsymbol{\nabla} a(\boldsymbol{r}_\perp) \times \hat{\boldsymbol{z}}] = -\frac{qN}{mcV}\oint \boldsymbol{\nabla} a \cdot d\boldsymbol{l} \equiv 0 \quad (5)$$

for any scalar function $a(x, y)$. Here $\oint$ denotes the closed-path integral in the transverse plane traced by $\boldsymbol{r}_\perp(t)$, and the line element $d\boldsymbol{l}(\boldsymbol{X}_Q(t)) = d\tau\, dX_Q/d\tau$ lies on that curve. Note that the above result would be obtained for *any* trajectory that closes in the transverse plane and the QLT trajectory is just a special case. The remaining contribution from the partial cycle in equation (4) has a maximum value of $M = 2\pi\tilde{b}/B_0$, where $\tilde{b}$ is a finite characteristic value of the perturbed magnetic field. This upper bound for $\Delta\mu(t)$ has been evaluated for one trajectory, but it holds equally well for all the elements of the ensemble, which may differ in starting positions or initial gyrophases. Therefore, the pitch-angle scattering coefficient is bounded by $\tilde{\phi} = \lim_{\Delta t\to\infty} M^2/\Delta t$. Taking the limit as $\Delta t \to \infty$, it follows that the contribution to pitch-angle scattering from two-dimensional magnetostatic turbulence in QLT vanishes.

The above result is anticipated in the literature but has not been stated explicitly, to our knowledge. For example, it is implicit in the dynamical QLT calculations of Bieber et al. (1994; eqs. [20] and [21] there) when the dynamical turbulence parameter is taken to zero, $\alpha \to 0$, recovering the magnetostatic limit. M. Forman (2002, private communication) has also proved the analogous result for axisymmetric two-dimensional turbulence. The above proof shows that the quasi-linear pitch-angle scattering rate $\phi_{QLT}$ is precisely zero for magnetostatic two-dimensional turbulence. However, we will now demonstrate using numerical simulations that two-dimensional turbulence makes a nonnegligible contribution to parallel scattering at large amplitude (see also Qin et al. 2002b; Shalchi et al. 2004).

## 2. NUMERICAL METHODS

We use numerical methods similar to Mace et al. (2000) and Qin et al. (2002a, 2002b) to follow trajectories of test particles and, upon averaging, determine numerical values of diffusion coefficients. Test particles obey the Newton-Lorentz equations, $m\, d\boldsymbol{V}(t)/dt = q\boldsymbol{V}(t)/c \times \boldsymbol{B}(\boldsymbol{x})$ and $d\boldsymbol{X}/dt = \boldsymbol{V}$, for particle velocity $\boldsymbol{V}(t)$ and position $\boldsymbol{X}(t)$, where $q$ is the particle charge. These are solved numerically using 64 bit arithmetic and a fourth-order Runge-Kutta method, employing a variable step size and a fifth-order error estimate to control relative error at a level of 1 part in $10^9$ at each time step. With this method, contributions to the spatial diffusion of particles due to the numerical scheme are negligible; detailed tests of the method are given in Qin (2002). We neglect laboratory-frame electric field and use as a model of the total static magnetic field $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}_0 + \boldsymbol{b}(\boldsymbol{x})$. Here $\boldsymbol{B}_0$ is the average and $\boldsymbol{b}$ the fluctuation, as in § 1; however, now we allow that $\boldsymbol{b}$ may consist of a two-dimensional component $\boldsymbol{b}^{2D}(x, y)$ and one-dimensional slab component $\boldsymbol{b}^{slab}(z)$. We vary the ratio of energy in the slab component $E^{slab}$ to the energy in the two-dimensional component $E^{2D}$, so that $E^{slab}/(E^{slab} + E^{2D})$ varies from 0 to 1; these limits correspond to purely two-dimensional and purely slab, respectively. For the two-dimensional component, the size of the simulation box is $100\lambda \times 100\lambda$, consisting of $N_x \times N_y$ points with $N_x = N_y = 4096$. The two-dimensional spectrum has the form

$$S_{xx}^{2D}(k_\perp) = \frac{C(\nu)\lambda\langle b_{2D}^2\rangle}{\pi k_\perp(1 + k_\perp^2\lambda^2)^\nu}; \quad (6)$$

here $\lambda$ is a scale that determines the bend-over wavenumber of the two-dimensional fluctuations, $C(\nu) = (2\pi^{1/2})^{-1}\Gamma(\nu)/\Gamma(\nu - \frac{1}{2})$, where $\Gamma$ is the gamma function and $\nu = \frac{5}{6}$. The slab box size is $100{,}000\lambda$ with $N_z = 2^{22}$ points, and its spectrum is

$$S_{xx}^{slab}(k_\parallel) = \frac{C(\nu)\lambda_z\langle b_{slab}^2\rangle}{(1 + k_\parallel^2\lambda_z^2)^\nu}, \quad (7)$$

where the parallel turbulence scale $\lambda_z = 10\lambda$ is on the order of the parallel correlation length. This is motivated by experimental results such as those of Robinson & Rusbridge (1971). Note that for these parameters, the ratio of two-dimensional to slab energy varies with wavenumber and can depart substantially from the global ratio of energies. In the simulation, all lengths are scaled to the perpendicular turbulence scale $\lambda$ as length scale.

## 3. SIMULATION RESULTS

We examine the contribution of the two-dimensional component of turbulence to parallel diffusive transport by calculating a parallel running diffusion coefficient $D_z(t) = \frac{1}{2} d\langle(\Delta z)^2\rangle(t)/dt$, where the derivative is computed using a suitable finite-difference procedure. Figure 1 shows $D_z(t)$ for pure two-dimensional ($E^{slab} = 0$) fluctuations with ratio of gyroradius to perpendicular correlation scale $r_L/\lambda = 0.05$ and large turbulence level $b/B_0 = 1$. At short times, $Vt/\lambda < 50$, the par-

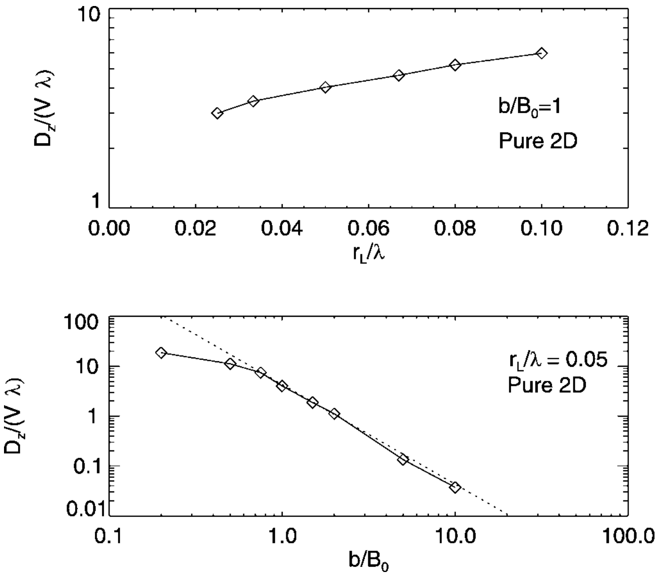


FIG. 2.—Numerically determined parallel diffusion coefficients from a series of test-particle simulations in pure two-dimensional fluctuations. *Top*, $b/B_0 = 1.0$, while $r_L/\lambda$ varies from 0.025 to 0.1; *bottom*, $r_L/\lambda = 0.05$, while $b/B_0$ varies from 0.2 to 10. The dotted line corresponds to $D_z \sim (b/B_0)^{-2}$.

allel displacements are free-streaming and nondiffusive, and there is an associated rapid rise of the running diffusion coefficient. A plateau is reached after a time at which most particles have traveled a few mean free paths, and thereafter the running diffusion coefficient remains approximately constant for very long times ($Vt/\lambda > 4000$). This constant may be identified as the parallel diffusion coefficient $\kappa_\parallel$, but for clarity we will continue to refer to the stable numerically determined values as $D_z$. This demonstrates that in pure two-dimensional magnetic turbulence with the given conditions, significant parallel diffusive transport is established.

Figure 2 shows numerically determined parallel diffusion coefficients in pure two-dimensional turbulence for several choices of parameters. The top panel shows parallel diffusion coefficients versus the dimensionless rigidity $r_L/\lambda$. The fluctuation level is constant, $b/B_0 = 1$. One can see that $D_z$ increases as the particle energy ($r_L/\lambda$) increases.

The bottom panel of Figure 2 shows $D_z$ versus turbulence level $b/B_0$ in pure two-dimensional turbulence with constant ratio $r_L/\lambda = 0.05$. As $b/B_0$ varies from a low (0.2) to a high (>1.0) value, $D_z$ decreases, that is, there is more parallel scattering. Furthermore, for strong turbulence ($b/B_0 > 0.6$), we find that $D_z \sim (b/B_0)^{-2}$ (*dotted line*).

We now examine how $D_z$ varies for different ratios of $E^{\rm slab}$ to $E^{\rm total}$. Figure 3 shows some results at constant strong turbulence level $b/B_0 = 1$ and fixed low particle energy, $r_L/\lambda = 0.05$. For reference we also show a theoretical value for the parallel diffusion coefficient from quasi-linear parallel scattering theory for a power-law inertial range extending to infinite wavenumber and using the slab power only (Jokipii 1966; Bieber et al. 1995). The parallel diffusion coefficient from simulation results and the QLT result are quite different, the numerical $D_z$ decreasing from a finite value at $E^{\rm slab}:E^{\rm total} = 0$ (purely two-dimensional) to a minimum value at around $E^{\rm slab}:E^{\rm total} = 0.5$ and then increasing to cross the QLT curve, attaining a local maximum value at around $E^{\rm slab}:E^{\rm total} = 0.9999$ (nearly pure-slab). From the cases shown and other, similar results that are not shown, we conclude that when the two-dimensional component is of sufficiently large amplitude, it

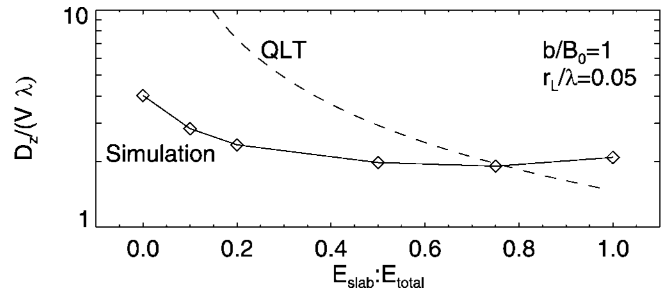


FIG. 3.—Parallel diffusion coefficients vs. $E^{\rm slab}:E^{\rm total}$ with constant strong turbulence level $b/B_0 = 1$ and low particle energy $r_L/\lambda = 0.05$. $E^{\rm slab}:E^{\rm total}$ ranges from 0 (purely two-dimensional) to 0.9999 (nearly pure-slab). The solid line is from the simulation, and the dashed line corresponds to QLT.

makes a substantial contribution to parallel diffusive transport and QLT is inaccurate, in that it assigns no contribution to scattering from the two-dimensional component.

We remark in passing that the discrepancy between QLT and the numerical results in Figure 3 for the nearly pure-slab case is not of great concern. The cases shown above have a large resonance gap associated with the numerical spectral cutoff. When we carry out a run with similar parameters but higher particle energy (not shown), $r_L/\lambda = 0.2$ (or $r_L/\lambda_z = 0.02$), there is a much smaller spectral gap, and we find that the parallel diffusion coefficient agrees much better with QLT in the slab limit. In addition, when comparing these results with applications such as heliospheric scattering, it is important to recall that dynamical effects, not considered here, are likely to be significant, especially for low-energy particles.

## 4. DISCUSSION

Previously it had been assumed, or shown only for special cases (i.e., axisymmetry), that in the QLT approach two-dimensional turbulence does not contribute to the parallel scattering rate. Here we have shown that this is a rigorous trait of all transverse two-dimensional turbulence models in QLT. Therefore, in a two-component, slab plus two-dimensional parameterization of anisotropic magnetostatic turbulence, only the slab component can contribute to a particle's parallel diffusive transport, in the context of QLT. However, the present numerical simulations demonstrate that the two-dimensional component of turbulence can make an important contribution to parallel diffusive transport. This result originally appeared in the first author's Ph.D. thesis (Qin 2002), and some of those results were reproduced in Shalchi et al. (2004).

A corollary to our proof is that a necessary condition for parallel scattering in two-dimensional turbulence is that the projection of the particle orbits onto the two-dimensional plane not close on themselves. Consequently, nonlinear (large $b/B_0$) orbit effects, or dynamical (time-varying $b$) effects, are essential, as these cause the projection of the orbit to return to a different transverse position after each gyroperiod $2\pi/\Omega$. On the other hand, if $b/B_0$ is small and static, the total magnetic field does not change very much during a gyroperiod, the orbits close on themselves, quasi-linear theory is still valid, and the scattering becomes vanishingly small. It seems likely that for the large-amplitude case for which the transverse positions are nonperiodic, there will be nonlinear perpendicular diffusion as well. Although this has not been demonstrated here, it is an effect apparently included in both the nonlinear guiding-center theory of perpendicular diffusion (Matthaeus et al. 2003) and in the weakly nonlinear theory in which perpendicular and parallel transport are determined simultaneously (Shalchi et al. 2004).

Evidently there are a variety of factors needed to fully describe particle scattering in large-amplitude turbulence, and we will not attempt to further examine these here, or even to catalog the possibilities. It is, however, certain that classical quasi-linear theory cannot explain the full range of large-amplitude scattering effects, as we have shown both that two-dimensional turbulence contributes significantly at large amplitudes and that magnetostatic QLT predicts a null result for that case. These results again confirm that either nonlinear (Völk 1973; Shalchi et al. 2004) or dynamical (Bieber et al. 1994) effects are needed to establish a realistic scattering theory. Finally, the present results again raise the issue of understanding the observed mean free paths of low-energy cosmic rays. The problem is to reconcile the observed mean free paths of solar energetic particles (as compiled, e.g., in the Palmer [1982] consensus data; see also Dröge 2003) with the observed turbulence levels, which appear, using QLT with a pure-slab model, to imply about an order of magnitude more scattering. For some time an acceptable resolution of this issue (Bieber et al. 1994) has seemed to be that 80%–90% of the turbulence energy resides in two-dimensional modes (or highly oblique, nearly two-dimensional modes) that contribute essentially nothing to the scattering rate. This theoretical explanation of the observations now must be reexamined, in view of the present results.

This research was supported in part by NSF grant ATM 00-00315, NASA grant NNG04GF81G, and National Natural Science Foundation of China grant 40325010.

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