Motion of Charged Particle in Fields

Plasma are complicated because motions of electrons and ions are determined by the electric and magnetic fields but also change the fields by the currents they carry.

For now we shall ignore the second part of the problem and assume that ***fields are prescribed.***

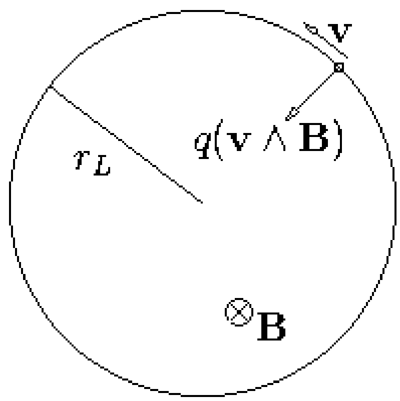
Even so, calculating the motions of a charged particle can be quite hard.

Equation of motion:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

v: velocity, q: charge, **E**: Electric Field, **B**: magnetic field

$q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$: Lorentz Force



Have to solve this differential equation, to get position (**r**) and velocity ($\mathbf{v}=\dot{\mathbf{r}}$) given **E**(**r**,t), **B**(**r**,t).

Approach: start simple, gradually generalize.

1. Uniform B field, E = 0

$$m\,\dot{v} = q\mathbf{v} \times \mathbf{B}$$

Qualitatively in the plane perpendicular to **B**: Acceleration is perpendicular to **v** so particle moves in a circle whose radius r_L is such as to satisfy

$$mr_L\Omega^2 = m\frac{v_\perp^2}{r_L} = |q|v_\perp B$$

Ω: angular (velocity) frequency

1st equality shows $\Omega^2 = \frac{v_\perp^2}{r_L^2}$ $\quad\quad(r_L = \frac{v_\perp}{\Omega})$

Hence second gives $m\frac{v_\perp}{\Omega}\Omega^2 = |q|v_\perp B$

i.e.

$$\Omega = \frac{|q|B}{m}$$

Particle moves in a circular orbit with angular velocity Ω ("Cyclotron frequency") and radius r_L ("Larmor Radius", "Gyro Radius").

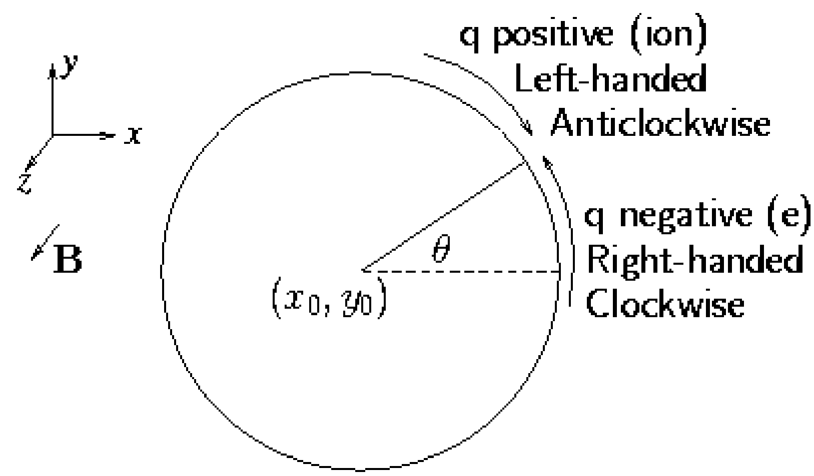
Some Vector Algebra:

- Particle Energy is constant. Proof : take **v**· Eq of motion then

$$m\dot{\mathbf{v}} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$$

- Parallel and perpendicular motions separate. $v_{//} = const.$ because accelerate is perpendicular to **B.**

Perpendicular Dynamics:
Take **B** in $\hat{z}$ direction and write components
$m\dot{v}_x = qv_yB, \; m\dot{v}_y = -qv_xB$
Hence $\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -(\frac{qB}{m})^2 v_x = -\Omega^2 v_x$
Solution: $v_x = v_\perp \cos\Omega t$ (choose zero of time)
Substitute back: $v_y = \frac{m}{qB}\dot{v}_x = -\frac{|q|}{q}v_\perp \sin\Omega t$
Integrate:
$x = x_0 + \frac{v_\perp}{\Omega}\sin\Omega t, \; y = y_0 + \frac{q}{|q|}\frac{v_\perp}{\Omega}\cos\Omega t$
This is the equation of a circle with center $r_0 = (x_0, y_0)$ and radius $r_L = \frac{v_\perp}{\Omega}$: Gyro Radius



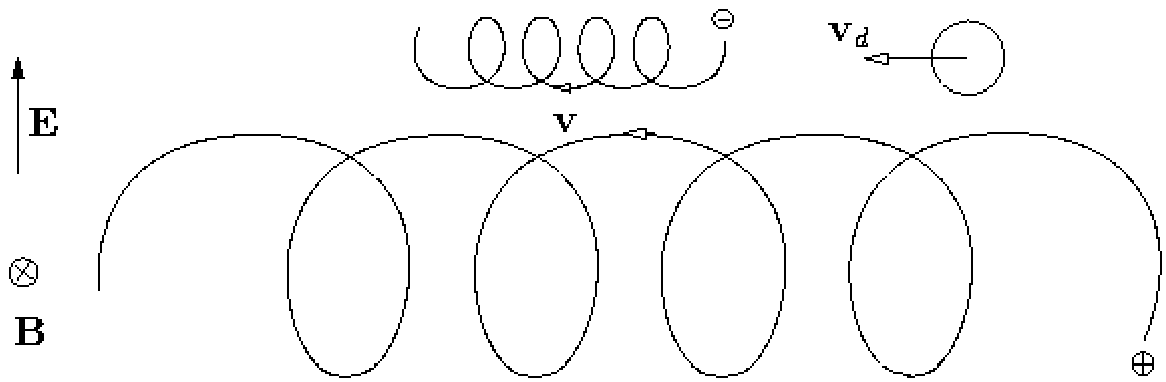
Direction of motion is as indicated opposite for opposite sign of charge:
Ions rotate anticlockwise
Electrons clockwise about the magnetic field.

The current carried by the plasma always is in such a direction as to ***reduce*** the magnetic field.
This is the property of a magnetic material which is “Diamagnetic”.
When $v_{//}$ is non-zero the total motion is along a
“helix”:

2. Uniform B and nonzero E

$$m\,\dot{v} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}$$

Parallel motion: Before, when $\mathbf{E} = 0$ this was $v_{//} = const.$ Now it is clearly $\dot{v}_{//} = \frac{qE_{//}}{m}$, constant acceleration along the field.
Perpendicular motion:
Qualitatively:



Speed of positive particle is greater at top than bottom so radius of curvature is greater. Result is that guiding center moves perpendicular to both **E** and **B**. It “drift” across the field.

Algebraically: It is clear that if we can find a constant velocity $\mathbf{v}_d$ that satisfies

$$\mathbf{E} + \mathbf{v}_d \times \mathbf{B} = 0$$

then the sum of this drift velocity plus the velocity $\mathbf{v}_L = \frac{d}{dt}[r_L e^{i\Omega(t-t_0)}]$
which we calculated for the $E = 0$ gyration will satisfy the equation of motion.
Take $\times\mathbf{B}$ the above equation:

$$0 = \mathbf{E} \times \mathbf{B} + (\mathbf{v}_d \times \mathbf{B}) \times \mathbf{B} = \mathbf{E} \times \mathbf{B} + (\mathbf{v}_d \cdot \mathbf{B})\mathbf{B} - B^2\mathbf{v}_d$$

so that

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

does satisfy it.
Hence the full solution is

$$\mathbf{v} = \mathbf{v}_{//} + \mathbf{v}_d + \mathbf{v}_L$$

where

$\dot{\mathbf{v}}_{//} = \frac{q\mathbf{E}_{//}}{m}$

and

$\mathbf{v}_d = \frac{\mathbf{E}\times\mathbf{B}}{B^2}$ is the "**E** × **B** drift" of the gyrocenter.

Comments on **E** × **B** drift:

1. It is independent of the properties of the drifting particle (q, m, v, whatever)
2. Hence it is in the same direction for electrons and ions.
3. Underlying physics for this is that in the frame moving at the **E** × **B** drift $\mathbf{E} = 0$. We have 'transformed away' the electric field.
4. Formula given above is exact except for the fact that relativistic effects have been ignored. They would important if $v_d \sim c$.

Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity, write **F** so that

$$m\dot{\mathbf{v}} = \mathbf{F} + q(\mathbf{v}\times\mathbf{B}) = q(\frac{1}{q}\mathbf{F} + \mathbf{v}\times\mathbf{B})$$

This is just like the Electric field case except with $\frac{1}{q}\mathbf{F}$ replacing **E**,

The drift is therefore:

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}$$

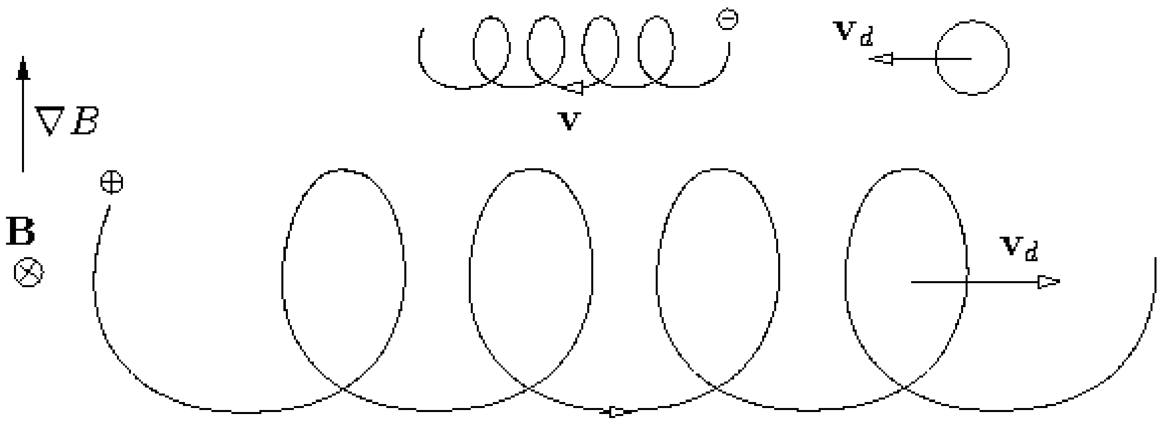
in this case, if force on electrons and ions is same, they drift in opposite directions.

This general formula can be used to get the drift velocity in some other cases of interest (see later).

3. Non-Uniform B Field

3.1 Grad-B drift

If B-lines are straight but the magnitude of B varies in space we get orbits that look qualitatively similar to the $\mathbf{E} \perp \mathbf{B}$ case:



Curvature of orbit is greater where B is greater causing loop to be small on the side. Result is a drift perpendicular to both **B** and ∇B. Notice, though, that electrons and ions go in opposite directions (unlike **E** × **B**).

Algebra:
We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v}_d + \mathbf{v}_L$, where $\mathbf{v}_d$ is constant.
We shall find that this can be done only approximately. Also we must have a simple expression for B. This we get by assuming that the Larmor radius is much smaller that the scale length of B variation. i.e. $r_L << B/|\nabla B|$ in which case we can express the field approximately as the first two terms in a Taylor expansion:

$$\mathbf{B} \simeq \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B}$$

Then substituting the decomposed velocity we get:

$$m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}_L = q(\mathbf{v} \times \mathbf{B}) = q[\mathbf{v}_L \times \mathbf{B}_0 + \mathbf{v}_d \times \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \times (\mathbf{r} \cdot \nabla)\mathbf{B}]$$

or

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{v}_d \times (\mathbf{r} \cdot \nabla)\mathbf{B}$$

Now we shall find that v_d/v_L is also small, like $r|\nabla B|/B$. Therefore the last term here is second order but the first two are first order. So we drop the last term.
Now the awkward part is that v_L and r_L are periodic. Substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} + \mathbf{v}_d \times (\mathbf{r}_0 \cdot \nabla)\mathbf{B}$$

We now average over a cyclotron period. The last term is $\propto \exp(-i\Omega t)$ so it averages to zero:

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + < \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} >$$

To perform the average use

$$\mathbf{r}_L = (x_L, y_L) = \frac{v_\perp}{\Omega}(\sin\Omega t, \frac{q}{|q|}\cos\Omega t)$$

$$\mathbf{v}_L = (\dot{x}_L, \dot{y}_L) = v_\perp(\cos\Omega t, -\frac{q}{|q|}\sin\Omega t)$$

So

$$[\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]_x = v_y y \frac{d}{dy} B$$

$$[\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]y = -v_x y \frac{d}{dy} B$$

(Taking ∇B to be in the y-direction)
Then

$$< v_y y >= - < \cos\Omega t \sin\Omega t > \frac{v_\perp^2}{\Omega} = 0$$

$$< v_x y >= \frac{q}{|q|} < \cos\Omega t \cos\Omega t > \frac{v_\perp^2}{\Omega} = \frac{1}{2}\frac{v_\perp^2}{\Omega}\frac{q}{|q|}$$

so

$$< \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} >= -\frac{1}{2}\frac{q}{|q|}\frac{v_\perp^2}{\Omega}\nabla B$$

Substitute in: $0 = \mathbf{v}_d \times \mathbf{B}_0 - \frac{1}{2}\frac{q}{|q|}\frac{v_\perp^2}{\Omega}\nabla B$ and solve as before to get:

$$\mathbf{v}_d = \frac{-\frac{1}{2}\frac{q}{|q|}\frac{v_\perp^2}{\Omega}\nabla B \times \mathbf{B}}{B^2} = \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\frac{\mathbf{B} \times \nabla B}{B^2}$$

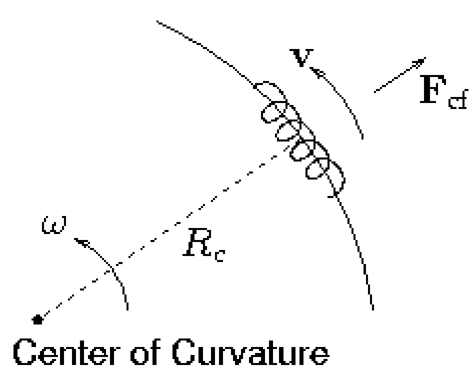
or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B} \times \nabla B}{B^2}$$

This is called the 'Grad-B drift'.

3.2 Curvature Drift

When the B-field lines are curved and the particle has a velocity $v_{//}$ along the field, another drift occurs.



Take |B| constant, radius of curvature R_e
To 1st order the particle just spirals along the field.
In the frame of the guiding center, a force appears because the frame is rotating about the center of curvature.

This centrifugal force is F_{cf}

$$F_{cf} = m\frac{v_{//}^2}{R_c}$$

pointing outward as a vector

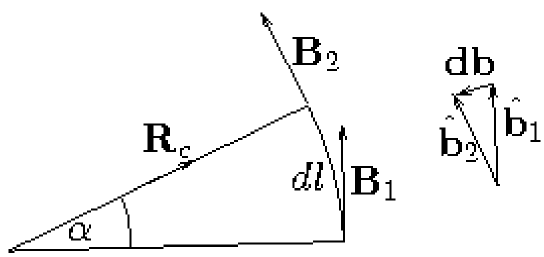
$$\mathbf{F}_{cf} = mv_{//}^2\frac{\mathbf{R}_c}{R_c^2}$$

[There is also a coriolis force $2m(\boldsymbol{\omega} \times \mathbf{v})$ but this averages to zero over a gyroperiod]
Use the previous formula for a force

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}_{cf} \times \mathbf{B}}{B^2} = \frac{mv_{//}^2}{qB^2}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2}$$

This is the "Curvature Drift".

It is often convenient to have this expressed in terms of the field gradients. So we relate $\mathbf{R}_c$ to ∇B etc. as follows:



(Caveats denote unit vectors)
From the diagram

$d\mathbf{b} = \hat{b}_2 - \hat{b}_1 = -\hat{R}_c\alpha$

and

$dl = \alpha R_c$

So

$\frac{d\mathbf{b}}{dl} = -\frac{\hat{R}_c}{R_c} = -\frac{\mathbf{R}_c}{R_c^2}$

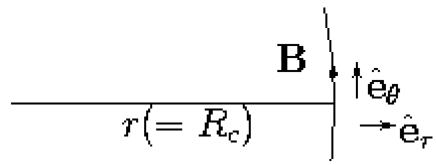
But (by definition) $\frac{db}{dl} = (\hat{b} \cdot \nabla)\hat{b}$

So the curvature drift can be written

$$\mathbf{v}_d = \frac{mv_{//}^2}{q}\frac{\mathbf{R}_c}{R_c^2} \times \frac{\mathbf{B}}{B^2} = \frac{mv_{//}^2}{q}\frac{\mathbf{B} \times (\hat{b} \cdot \nabla)\hat{b}}{B^2}$$

3.3 Vacuum Fields: Relation between ∇B & R_c drifts

The curvature and ∇B are related because of Maxwell's equations, their relation depends on the current density j. A particular case of interest is $j = 0$: vacuum fields.



$\nabla \times \mathbf{B} = 0$ (static case)
Consider the z-component
$0 = (\nabla \times \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)$ ($B_r = 0$ by choice)
$= \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}$
or, in other words, $(\nabla B)_r = -\frac{B}{R_c}$
[Note also $0 = (\nabla \times \mathbf{B})_\theta = \frac{\partial}{\partial z}B_\theta$: $(\nabla B)_z = 0$]
and hence $(\nabla B)_{perp} = -B\mathbf{R}_c/R_c^2$
Thus the grad B drift can be written:

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B} \times \nabla B}{B^3} = \frac{mv_\perp^2}{2q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}$$

and the total drift across a vacuum field becomes

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}(mv_{//}^2 + \frac{1}{2}mv_\perp^2)\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}$$

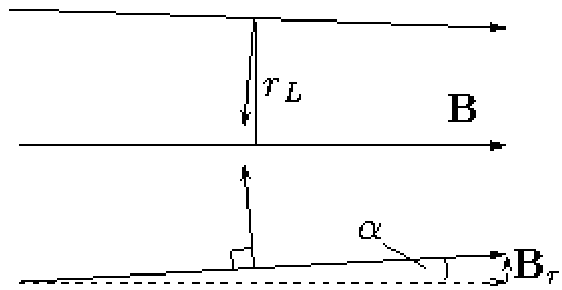
Notice the following:

1. R_c & ∇B drifts are the same direction.
2. They are in opposite directions for opposite charges.
3. They are proportional to particle energies.
4. Curvature ↔ Parallel energy ($\times 2$), ∇B ↔ Perpendicular energy
5. As a result one can very quickly calculate the average drift over a thermal distribution of particles because $< \frac{1}{2}mv_{//}^2 >= T/2, < \frac{1}{2}mv_\perp^2 >= T$ (2 degrees of freedom)

Therefore

$$< \mathbf{v}_R + \mathbf{v}_{\nabla B} >= \frac{2T}{q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}\left(= \frac{2T}{q}\frac{\mathbf{B} \times (\hat{b} \cdot \nabla)\hat{b}}{B^2}\right)$$

4. The Effects of Parallel Field Gradients Mirror Effect: $\boldsymbol{E} = \mathbf{0}$, $\nabla \boldsymbol{B}$ // $\boldsymbol{B}$



In the above situation there is a net force along B. Force is

$$< F_{//} > -|q\mathbf{v} \times \mathbf{B}| \sin\alpha = -|q| v_\perp B \sin\alpha$$

$$\sin\alpha = -B_r/B$$

Calculate B_r as function of B_z from $\nabla \cdot \mathbf{B} = \mathbf{0}$ $(B_\theta = 0)$

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0$$

Hence

$$rB_r = -\int r\frac{\partial B_z}{\partial z}dr$$

Suppose r_L is small enough that $\frac{\partial B_z}{\partial z} \simeq const.$

$$[rB_r]_0^{r_L} = -\int_0^{r_L} rdr\frac{\partial B_z}{\partial z} = -\frac{1}{2}r_L^2\frac{\partial B_z}{\partial z}$$

So

$$B_r(r_L) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z}$$

$$\sin\alpha = -\frac{B_r}{B} = \frac{r_L}{2}\frac{1}{B}\frac{\partial B_z}{\partial z}$$

Hence

$$< F_{//} >= -|q|\frac{v_\perp r_L}{2}\frac{\partial B_z}{\partial z} = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\partial B_z}{\partial z}$$

As particle enters increasing field region it experiences a net parallel retarding force.
Define Magnetic Moment

$$\mu \equiv \frac{\frac{1}{2}mv_\perp^2}{B}$$

Note this is consistent loop current definition:

$$\mu = AI = \pi r^2 \cdot \frac{|q|v_\perp}{2\pi r_L} = \frac{|q|r_L v_\perp}{2}$$

force is

$$F_{//} = -\mu\frac{\partial B_z}{\partial z}$$

This is force on a 'magnetic dipole' of moment μ

$$F_{//} = \boldsymbol{\mu} \cdot \nabla_{//}\mathbf{B}$$

Our μ always points along **B** but in opposite direction

Force on an Elementary Magnetic Moment Circuit

Consider a plane rectangular circuit carrying current I having elementary area $dxdy = dA$. Regard this as a vector pointing in ẑ direction **dA.**
The force on this circuit in a field **B**(**r**) is **F** such that

$$\begin{aligned} F_x &= Idy[B_z(x+dx) - B_z(x)] \quad = Idydx\frac{\partial B_z}{\partial x} \\ F_y &= -Idx[B_z(y+dy) - B_z(y)] \quad = Idydx\frac{\partial B_z}{\partial y} \\ F_z &= -Idx[B_y(y+dy) - B_y(y)] \; - Idy[B_x(x+dx) - B_x(x)] \\ &= -Idxdy[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}] = Idydx\frac{\partial B_z}{\partial z} \end{aligned}$$

(using $\nabla \cdot \mathbf{B} = \mathbf{0}$)

Hence, summarizing: $\mathbf{F} = Idydx\nabla B_z$
Now define $\boldsymbol{\mu} = Id\mathbf{A} = Idydx\ \hat{z}$
and take it constant. Then clearly the force can be written

$$\mathbf{F} = \nabla(\mathbf{B} \cdot \mathbf{u}) = (\nabla B) \cdot \boldsymbol{\mu}$$

$\boldsymbol{\mu}$ is the (vector) magnetic moment of the circuit.
The shape of the circuit does not matter since any circuit can be consider to be composed of the sum of many rectangular circuits. So in general

$$\boldsymbol{\mu} = Id\mathbf{A}$$

and the force is

$$\mathbf{F} = \nabla(\mathbf{B} \cdot \boldsymbol{\mu})$$

(μ constant)

We shall show in a moment that $|\boldsymbol{\mu}|$ is constant for a circulating particle, regarded as an elementary circuit. Also, $\boldsymbol{\mu}$ for a particle always points in the -**B** direction. [Note that this means that the effect of particles on the field is to decrease it.] Hence the force may be written

$$\mathbf{F} = -\mu\nabla B$$

This gives us both:

- Magnetic Mirror Force: $F_{//} = -\mu\nabla_{//}B$
- Grad B Drift: $v_{\nabla B} = \frac{1}{q}\frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2}$

μ is a constant of the motion

“Adiabatic Invariant”
Proof from $F_{//}$
Parallel equation of motion

$$m\frac{dv_{//}}{dt} = F_{//} = -\mu\frac{dB}{dz}$$

So

$$mv_{//}\frac{dv_{//}}{dt} = \frac{d}{dt}(\frac{1}{2}mv_{//}^2) = -\mu v_z\frac{dB}{dz} = -\mu\frac{dB}{dt}$$

or

$$\frac{d}{dt}(\frac{1}{2}mv_{//}^2) + \mu\frac{dB}{dt} = 0$$

Conservation of Total KE

$$\frac{d}{dt}(\frac{1}{2}mv_{//}^2 + \frac{1}{2}mv_{\perp}^2) = 0$$

$$\frac{d}{dt}(\frac{1}{2}mv_{//}^2 + \mu B) = 0$$

Combine

$$\frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = 0$$

$$\frac{du}{dt} = 0$$

As required.

Angular Momentum
of particle about the guiding center is

$$r_L mv_{\perp} = \frac{mv_{\perp}}{|q|B}mv_{\perp} = \frac{2m}{|q|}\frac{1/2mv_{\perp}^2}{B} = \frac{2m}{|q|}\mu$$

Conservation of magnetic moment is basically conservation of angular momentum about the guiding center.

Proof direct from Angular Momentum
Angular momentum about the guiding center is conserved because θ is (locally) ignorable. However it is canonical angular momentum that is conserved.

$$p = [\mathbf{r} \times (m\mathbf{v} + q\mathbf{A})]_z$$

Here **A** is the vector potential such that $\mathbf{B} = \nabla \times \mathbf{A}$
The definition of the vector potential means that

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$$

Integrate over $r = 0$ to r_L

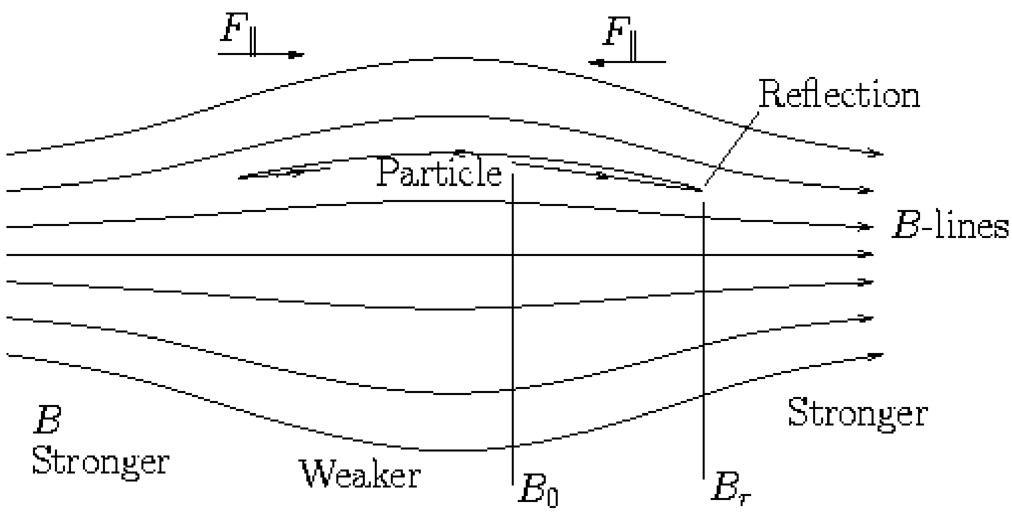
$$r_L A_\theta(r_L) = \int_0^{r_L} rB_z dr = \frac{r_L^2}{2}B_z = \frac{1}{2}(\frac{mv_{\perp}}{|q|B})^2 B_z = \frac{\mu m}{|q|^2}$$

Then

$$p = -\frac{q}{|q|}r_L m v_\perp + q\frac{\mu m}{|q|^2}$$
$$= -\frac{q}{|q|}\frac{mv_\perp}{|q|B}mv_\perp + \frac{m}{q}\mu$$
$$= \frac{m}{q}(-2\mu + \mu) = -\frac{m}{q}\mu$$

Thus if p is constant, i.e. canonical momentum conserved, then $\mu = const.$
Conservation of μ is basically conservation of angular momentum of particle about G.C.

Mirror Trapping



$F_{//}$ may be enough to reflect particle back. But may not!
Let's calculate whether it will:
Suppose reflection occurs.
At reflection point $v_{//r} = 0$
Energy Conservation $\frac{1}{2}m(v_{\perp 0}^2 + v_{//0}^2) = \frac{1}{2}mv_{\perp r}$
μ conservation $1/2mv_{\perp 0}^2/B_0 = 1/2mv_{\perp r}^2/B_r$
Hence $v_{\perp 0}^2 + v_{//0}^2 = B_r/B_0 v_{\perp 0}$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{//0}^2}$$

Pitch Angle θ

$$\tan\theta = \frac{v_\perp}{v_{//}}$$

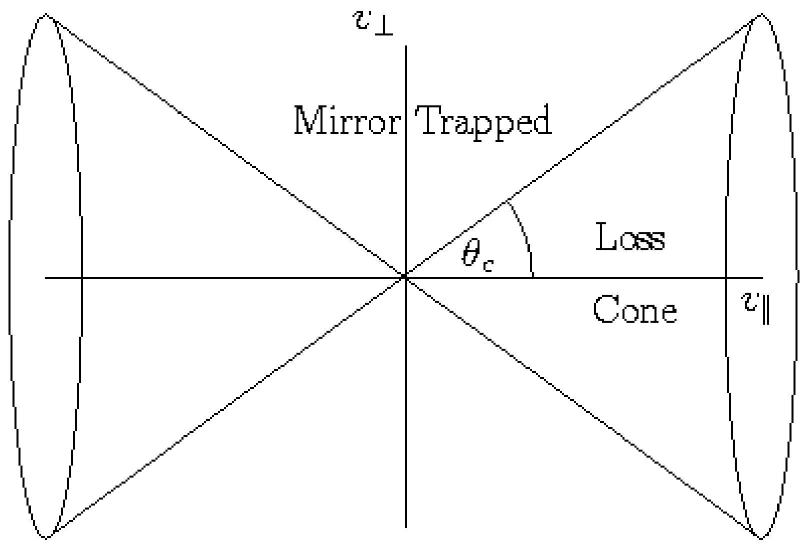
$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{//0}^2} = \sin^2\theta$$

So given a pitch angle θ_0 reflection takes place where $B_0/B_r = \sin^2\theta_0$
If θ_0 is too small no reflection can occur.

Critical ang θ_c is obviously

$$\theta_c = \sin^{-1}(B_0/B_1)^{1/2}$$

Loss Cone is all $\theta << \theta_c$



Important of Mirror Ratio: $R_m = B_1/B_0$.

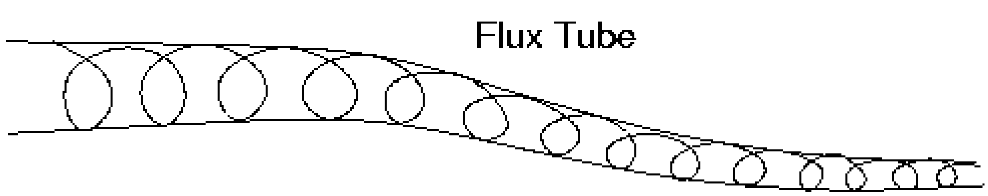
<u>Other Features of Mirror motions</u>

Flux enclosed by gyro orbit is constant.

$$\Phi = \pi r_L^2 B = \pi \frac{m^2 v_\perp^2}{q^2 B^2} B$$

$$= \frac{2\pi m}{q^2} \frac{\frac{1}{2} m v_\perp^2}{B}$$

$$= \frac{2\pi m}{q^2} \mu = const.$$

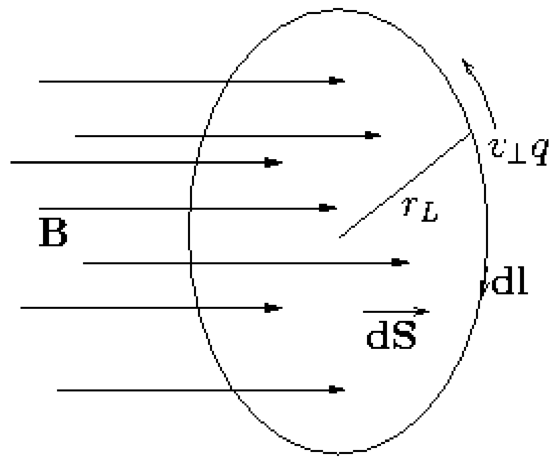


Note that if B changes 'suddenly'. μ might not be conserved.
Basic requirement

$$r_L << B/|\nabla B|$$

Slow variation of B (relative to r_L).

5. Time Varying B Field (E inductive)



Particle can gain energy from the inductive E field $\nabla \times \mathbf{E} = -\partial B/\partial t$
or

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int_s \dot{\mathbf{B}} \cdot d\mathbf{S} = -\frac{d\Phi}{dt}$$

Hence work done on particle in 1 revolution is

$$\delta w = -\oint |q| \mathbf{E} \cdot d\mathbf{l} = |q| \int_s \dot{\mathbf{B}} \cdot d\mathbf{S} = |q| \frac{d\Phi}{dt} = |q| \dot{B} \pi r_L^2$$

(dl and $v_\perp q$ are in opposite directions)

$$\delta(\frac{1}{2} m v_\perp^2) = |q| \dot{B} \pi r_L^2 = \frac{2\pi \dot{B} m}{|q| B} \frac{\frac{1}{2} m v_\perp^2}{B}$$

$$= \frac{2\pi \dot{B}}{|\Omega|} \mu$$

Hence

$$\frac{d}{dt}(\frac{1}{2} m v_\perp^2) = \frac{|\Omega|}{2\pi} \delta(\frac{1}{2} m v_\perp^2) = \mu \frac{dB}{dt}$$

$$\frac{d}{dt}(\frac{1}{2} m v_\perp^2) = \frac{d}{dt}(\mu B)$$

Hence

$$\frac{d\mu}{dt} = 0$$

Notice that since $\Phi = 2\pi m / q^2 \; \mu$
this is just another way of saying that the flux through the gyro obit is conserved.
Notice also energy increase. Method of 'heating'. Adiabatic Compression.

6. Time Varying E-field (E, B uniform)

Recall the $E \times B$ drift: $\mathbf{v}_{\mathbf{E} \times \mathbf{B}} = \mathbf{E} \times \mathbf{B} / B^2$

When E various so dose $\mathbf{v}_{\mathbf{E}\times\mathbf{B}}$. Thus the guiding center experiences an acceleration

$$\dot{\mathbf{v}}_{\mathbf{E}\times\mathbf{B}} = \frac{d}{dt}(\mathbf{E}\times\mathbf{B}/B^2)$$

In the frame of the guiding center which is accelerating, a force if felt.

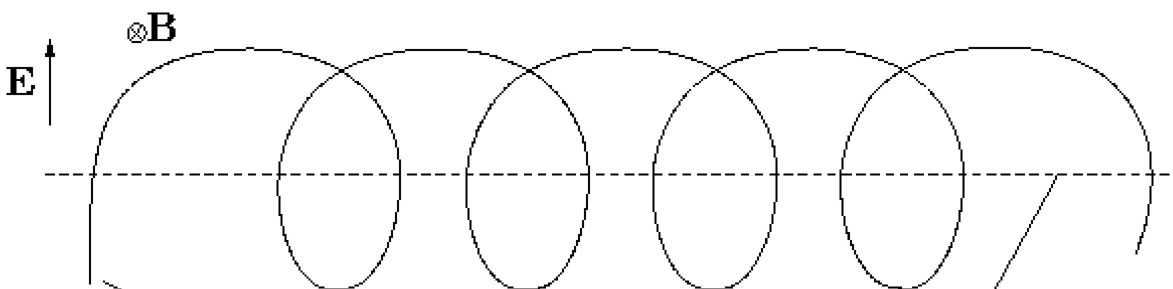
$$\mathbf{F}_a = -m\frac{d}{dt}\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)$$

(Pushed back into seat! -ve.)
This force produces another drift

$$\begin{aligned}\mathbf{v}_p &= \frac{1}{q}\frac{\mathbf{F}_a\times\mathbf{B}}{B^2} = -\frac{m}{qB^2}\frac{d}{dt}\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\times\mathbf{B} \\ &= -\frac{m}{qB}\frac{d}{dt}((\mathbf{E}\cdot\mathbf{B})\mathbf{B} - B^2\mathbf{E}) \\ &= \frac{m}{qB^2}\dot{\mathbf{E}}_\perp\end{aligned}$$

This is called the “polarization drift”

$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_{E\times B} + \mathbf{v}_p = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{m}{qB^2}\dot{\mathbf{E}}_\perp \\ &= \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp\end{aligned}$$



Start-up effect: When we “switch on” an electric field the average position (gyro center) of an initially stationary particle shifts over by ~1/2 the orbit size. The polarization drift is this polarization effect on the medium.
Total shift due to v_p is

$$\Delta\mathbf{r} = \int \mathbf{v}_p dt = \frac{m}{qB^2}\int\dot{\mathbf{E}}_\perp = \frac{m}{qB^2}[\Delta\mathbf{E}_\perp]$$

<u>Direct Derivation of $\frac{dE}{dt}$ effect: “Polarization Drift”</u>

Consider an oscillatory field. $E = Ee^{-i\omega t}$

$$\begin{aligned}m\frac{d\mathbf{v}}{dt} &= q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \\ &= q(\mathbf{E}e^{-i\omega t} + \mathbf{v}\times\mathbf{B})\end{aligned}$$

Try for a solution in the form

$$\mathbf{v} = \mathbf{v}_D e^{-i\omega t} + \mathbf{v}_L$$

where, as usual, $\mathbf{v}_L$ satisfies $m\dot{\mathbf{v}}_L = q\mathbf{v}_L \times \mathbf{B}$
Then
(1)

$$m(-i\omega)\mathbf{v}_D = q(\mathbf{E} + \mathbf{v}_D \times \mathbf{B})$$

solve for $\mathbf{v}_D$: Take $\times\mathbf{B}$ This equation:
(2)

$$-mi\omega(\mathbf{v}_D \times \mathbf{B}) = \mathbf{q}(\mathbf{E} \times \mathbf{B} + (\mathbf{B} \cdot \mathbf{v}_D)\mathbf{B} - B^2\mathbf{v}_D$$

add $mi\omega \times (1)$ to $q \times (2)$ to eliminate $\mathbf{v}_D \times \mathbf{B}$

$$m^2\omega^2\mathbf{v}_D + q^2(\mathbf{E} \times \mathbf{B} - B^2\mathbf{v}_D) = mi\omega q\mathbf{E}$$

or

$$\mathbf{v}_D[1 - \frac{m^2\omega^2}{q^2B^2}] = -\frac{mi\omega}{qB^2}\mathbf{E} + \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

i.e.

$$\mathbf{v}_D[1 - \frac{\omega^2}{\Omega^2}] = -\frac{i\omega q}{\Omega B|q|}\mathbf{E} + \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

since $-i\omega \leftrightarrow \partial/\partial t$ this is the same formula as we had before: the sum of polarization and $E \times B$ drifts except for the $[1 - \frac{\omega^2}{\Omega^2}]$ term. This term comes from the change in $\mathbf{v}_D$ with time (accel). Thus our earlier expression was only approximate. A good approx if $\omega \ll \Omega$.

7. Non Uniform E (Finite Larmor Radius)

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E}(\mathbf{r}) + \mathbf{v} \times \mathbf{B})$$

Seek the usual solution $\mathbf{v} = \mathbf{v}_D + \mathbf{v}_g$
Then average out over a gyro orbit

$$< m\frac{d\mathbf{v}_D}{dt} >= 0 =< q(\mathbf{E}(\mathbf{r}) + \mathbf{v} \times \mathbf{B}) >$$

$$= q[< \mathbf{E}(\mathbf{r}) > + \mathbf{v}_D \times \mathbf{B}]$$

Hence drift is obviously

$$\mathbf{v}_D = \frac{< \mathbf{E}(\mathbf{r}) > \times \mathbf{B}}{B^2}$$

So we just need to find the average E field experienced.
Expand E as a Taylor series about the G.C.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + (\mathbf{r} \cdot \nabla)\mathbf{E} + (\frac{x^2}{2!}\frac{\partial^2}{\partial x^2} + \frac{y^2}{2!}\frac{\partial^2}{\partial y^2})\mathbf{E} + cross\_terms + \ldots$$

(e.g. cross terms are $xy\frac{\partial^2}{\partial x \partial y}\mathbf{E}$)
Average over a gyro orbit: $\mathbf{r} = r_L(\cos\theta, \sin\theta, 0)$

Average of cross terms = 0.
Then

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + (\mathbf{r} \cdot \nabla)\mathbf{E} + \frac{< r_L^2 >}{2 \cdot 2!}\nabla^2\mathbf{E}$$

linear term $< r_L >= 0$, So

$$< \mathbf{E}(\mathbf{r}) >\simeq \mathbf{E} + \frac{r_L^2}{4}\nabla^2\mathbf{E}$$

Hence $E \times B$ with 1st finite-larmor-radius correction is

$$\mathbf{v}_{E\times B} = [(1 + \frac{r_L^2}{4}\nabla^2)\mathbf{E}] \times \mathbf{B}/B^2$$

[Note: Grad B drift is a finite Larmor effect already.]

Second and Third Adiabatic Invariants

There are additional approximately conserved quantities like μ in some geometries.

Summary of Drifts

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad \text{Electric Field}$$

$$\mathbf{v}_F = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} \quad \text{General Force}$$

$$\mathbf{v}_E = (1 + \frac{r_L^2}{4}\nabla^2)\frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad \text{Nonuniform E}$$

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B} \times \nabla B}{B^3} \quad \text{Grad B}$$

$$\mathbf{v}_R = \frac{mv_{//}^2}{q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \quad \text{Curvature}$$

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}(mv_{//}^2 + \frac{mv_\perp^2}{2})\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \quad \text{Vacuum Fields}$$

$$\mathbf{v}_p = \frac{q}{|q|}\frac{\dot{\mathbf{E}}_\perp}{|\Omega|B} \quad \text{Polarization}$$

Mirror Motion:

$$\mu \equiv \frac{mv_\perp^2}{2B} \quad \text{is constant}$$

Force is

$$\mathbf{F} = -\mu\nabla B$$