Fluid Description of Plasma

The single particle approach gets to be horribly complicated, as we have seen. Basically we need a more statistical approach because we can't follow each particle separately. Fortunately, this is not usually necessary because, surprisingly, the majority of plasma phenomena observed in real experiments can be explained by a rather crude fluid model, in which the identity of the individual particle is neglected, and only the motion of fluid elements is taken into account. Of course, in the case of plasmas, the fluid contains electrical charges. In an ordinary fluid, frequent collisions between particles keep the particles in a fluid element moving togehter. It is surprising that such a model works for plasmas, which general have infrequent collisions.

In the fluid approximation, we consider the plasma to be composed of two or more interpenetrating fluids, one for each species. In the simplest case, when there is only one specis of ion, we shall need two equations of motion, one for the positively charged ion fluid and one for the negatively charged electron fluid. In a partially ionized gas, we shall also need an equatuon for the fluid of neutral atoms. The neutral fluid will interact with the ions and electrons only through collisions. The ion and electron fluids will interact with each other even in the absense of collisions, becuase of the E and B fields are generated.

4.1 Fluid Equation of Motion

A. **Equation of Motion – Neglecitng collisions and thermal motion**

(1) Equation of motion for a single particle: (velocity **v**)

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

All partilces in a fluid element will move together with average velocity **u,** because will neglect collisions and thermal effects. We can also take **u** = **v**, and equation of motion for fluid element of particle density n is:

$$nm\frac{d\mathbf{u}}{dt} = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

where $\frac{d}{dt}$ is to be taken *at the position of the particles* (fluid element) ⇒ not very convenient. We wish to have an equation for fluid elements *fixed in space*.

(2) Transform to variables in a fixed frame that move with fluid element.

To make the transformtion to variables in a fixed frame, consider **G**(x,t) to be any property of a fluid in one-dimensional x space. The change of G with time in a frame moving with the fluid is the sum of two terms:

$$\frac{d\mathbf{G}(x,t)}{dt} = \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x}\frac{dx}{dt} = \frac{\partial \mathbf{G}}{\partial t} + u_x\frac{\partial \mathbf{G}}{\partial x}$$

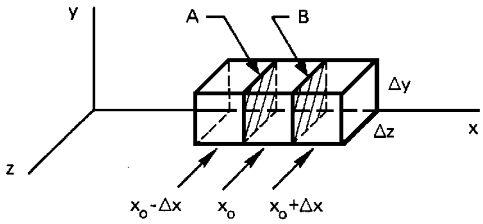
The first term on the right represents the change of G at a fixed point in space, and the second term represents the change of G as the observer moves with the fluid into a region in which G is different. In three dimensions

$$\frac{d\mathbf{G}}{dt} = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{G}$$

This is called the convective derivative.
In the case of a plasma, we take **G** to be the fluid velocity **u**,

$$nm[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

B. Equation of Motion – Including Thermal Effects (Pressure Term)
When thermal motions are taken into accout, a pressure force has to be added to the right-hand side. This force arised from the random motion of particles in and out of a fluid element and does not appear in the equation for a single particle. The random motion of the particles in the fluid element is described as a collective effect.



(1) Consider only the x-component of motion through Faces A and B of the fluid element if the figure, centered at (x_0, $\frac{1}{2}\Delta y$, $\frac{1}{2}\Delta z$).
The number of particles per second passing through the face A with velocity υ_x is

$$\Delta n_\upsilon \upsilon_x \Delta y \Delta z$$

where Δn_υ is the number of particles per m^3 with velocity υ_x :

$$\Delta n_\upsilon = \Delta \upsilon_x \iint f(\upsilon_x, \upsilon_y, \upsilon_z) d\upsilon_y d\upsilon_z$$

f is the "distribution function" of particles in velocity space at a particular spatial location.
Each particle carries a momentum $m\upsilon_x$. The momentum P_{A^+} carried into the element at x_0 the through A (The momentum through face A, from particles with $\upsilon_x > 0$, in the fluid elelment centered at ($x_0 - \Delta x$, $\frac{1}{2}\Delta y$, $\frac{1}{2}\Delta z$) is then

$$P_{A^+} = m\Delta y \Delta z \int_0^\infty \upsilon_x^2 f d\upsilon_x \iint_{-\infty}^{\infty} d\upsilon_y d\upsilon_z$$

For a properly normalized f we have

$$n = \iiint_{-\infty}^{\infty} f d\upsilon_x d\upsilon_y d\upsilon_z$$

And we can also define the average $< \upsilon_x^2 >$ as:

$$< \upsilon_x^2 > = \frac{\int_{-\infty}^{\infty} \upsilon_x^2 f d\upsilon_x \iint_{-\infty}^{\infty} d\upsilon_y d\upsilon_z}{\iiint_{-\infty}^{\infty} f d\upsilon_x d\upsilon_y d\upsilon_z}$$

(2) Then P_{A^+} can be written as

$$P_{A^+} = m\Delta y\Delta z \frac{1}{2} [n < \upsilon_x^2 >]_{x_0-\Delta x}$$

The factor $\frac{1}{2}$ comes from the fact that only half the particles in the cube at $x_0 - \Delta x$ are going toward face A. Similarly, the momentum carried out through face B is

$$P_{B^+} = m\Delta y\Delta z \frac{1}{2} [n < \upsilon_x^2 >]_{x_0}$$

(3) The net gain in momentum for particles with

$$\begin{aligned} P_{A^+} - P_{B^+} &= m\Delta y\Delta z \frac{1}{2} [(n < \upsilon_x^2 >)_{x_0-\Delta x} - (n < \upsilon_x^2 >)_{x_0}] \\ &= \frac{1}{2} m\Delta y\Delta z(-\Delta x) \frac{\partial}{\partial x} (n < \upsilon_x^2 >) \end{aligned}$$

This result will be just doubled by the contribution of left-moving particles, since they carry negative x momentum and also move in the opposite direction relative to the gradient of $n < \upsilon_x^2 >$. The total change of momentum of the fluid element at x_0 is therefore

$$\frac{\partial}{\partial t} (nmu_x)\Delta x\Delta y\Delta z = -m \frac{\partial}{\partial x} (n < \upsilon_x^2 >)\Delta x\Delta y\Delta z$$

(4) Let the velocity υ_x of a particle be decomposed into two parts,

$$\upsilon_x = u_x + \upsilon_{xr}, \quad u_x =< \upsilon_x >$$

u_x: x-component of fluid velocity
υ_{xr}: x-component of random thermal motion.
For a one-dimensional Maxwellian distribution, we have $\frac{1}{2} m < \upsilon_{xr}^2 >= \frac{1}{2} kT$

$$< \upsilon_x^2 >= u_x^2 + kT/m$$

$$\frac{\partial}{\partial t} (nmu_x) = -m \frac{\partial}{\partial x} [(u_x^2 + kT/m)n]$$

Re-grouping terms yield:

$$mn(\frac{\partial}{\partial t} u_x + u_x \frac{\partial}{\partial x} u_x) + mu_x(\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} nu_x) = - \frac{\partial}{\partial x} (nkT)$$

But $\frac{\partial}{\partial t} n + \frac{\partial}{\partial x} nu_x = 0$, from Continuity Equation (see section 4.2)

$$mn(\frac{\partial}{\partial t} u_x + u_x \frac{\partial}{\partial x} u_x) = \frac{\partial}{\partial x} p, \quad p = nkT = \text{pressure}$$

(5) This is the usual pressure-gradient force. Adding the electromagnetic

forces and generalizing to three dimensions, we have the fluid equation:

$$nm[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p$$

(6) What we have derived is only a special case: the transfer of x momentum by motion in the x direction; and we have assumed that the fluid is isotropic, so that the same result holds in the y and z directions. But it is possible to transfer y momentum by motion in the x direction. Suppose that the y-velocities of particles at $x_0 - \Delta x$ and $x_0 + \Delta x$ were a maximum, and that $v_y = 0$ at x_0. Then particles passing through Faces A and B would bring more Y-momentum into the fluid element at x_0 than they take out. This would give rise to a shear stress on the fluid element at x_0, which must be described in general by a stress tensor, $\overleftrightarrow{\mathbf{P}}$, whose components $P_{ij} = mn\overline{v_i v_j}$ specify both the direction of motion and the component of momentum involved. In the general case, the term $-\nabla p$ is repalced by $-\nabla \cdot \overleftrightarrow{P}$. The off-diagonal elements of $\overleftrightarrow{P}$ are usually associated with viscosity.

C. Equation of Motion – Including Collisions.

If a neutral gas is present, the charged fluid can exchange momentum with it through collisions. The momentum lost per collision will be proportional to the relative velocity **u - u**$_0$, where **u**$_0$ is the velocity of the neutral fluid. If τ, the mean free time between collisions, is approximately constant, the resulting force term can be roughly written as $-mn(\mathbf{u} - \mathbf{u}_0)/\tau$. Then the fluid equation of motion including neutral collisions and thermal effects is:

$$nm\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \overleftrightarrow{\mathbf{P}} - \frac{mn(\mathbf{u} - \mathbf{u}_0)}{\tau}$$

Notes:

(a) In the derivation, we did actually assume implicitly that there were collisions when we took the velocity distribution to be Maxwellian. Such a distribution generally comes about as the result of frequent collisions. However, this assumption was used only to take the averages of v_{xr}^2. Any other distribution with the same average would give us the same answer. The fluid theory, therefore, is not very sensitive to devviations from the Maxwellian distribution, although there are instances in which these deviations are important.

(b) Another reason the fluid model works for plasmas is that the magnetic field, when there is one, can play the role of collisons in a certain sense.

4.2 Fluid Equation of Continuity

Conservation of matter requires that the number of particles N in a volume V, can only change if there is net particle flux across the surface S bounding that volume,

$\frac{\partial N}{\partial t}$ ≡ change in particle number

$n\mathbf{u}$ ≡ particle flux

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\int_V ndV = -\int_S n\mathbf{u}\cdot d\mathbf{A} = -\int_V \nabla\cdot(n\mathbf{u})dV \quad \text{(by divergence theorem)}$$

Since this must hold for any volume V, the integrands must be equal:

$$\frac{\partial}{\partial t}n + \nabla\cdot(n\mathbf{u}) = 0 \qquad \text{equation of continuity}$$

4.3 Fluid Equation of State

One more relation is needed to close the system of equations. For this, we can use the thermodynamic equation of state relating p to n:

$$p = C\rho^{\gamma}$$

Where C is a constant and γ is the ratio of specfic heats C_p/C_v.

$$\nabla p = C\gamma\rho^{\gamma-1}\nabla\rho = \gamma p/\rho\nabla\rho \Rightarrow \frac{\nabla p}{p} = \gamma\frac{\nabla n}{n} \qquad (\rho = mn)$$

(1) For isothermal compression:$\nabla p = kT\nabla n \Rightarrow \gamma = 1$

(2)Adiabatic compression (T also changes)

$$\frac{\nabla n}{n} + \frac{\nabla T}{T} = \gamma\frac{\nabla n}{n} \Rightarrow \frac{\nabla T}{T} = (\gamma - 1)\frac{\nabla n}{n}$$

(3)More general (adiabatic), $\gamma = (2 + N)/N$,

where N is the number of degrees of freedom, it is valid for negligible heat flow.

4.4. Summary of Two-Fluid Equations:

For simplicity, let the plasma have only two species: ions and electrons; extension to more species is trival.

Species j (j=electrons, ions)

Plasma Response

1. Continuity:

$$\frac{\partial n_j}{\partial t} + \nabla\cdot(n_j\boldsymbol{\upsilon}_j) = 0$$

2. Momentum:

$$m_jn_j[\frac{\partial\boldsymbol{\upsilon}_j}{\partial t} + (\boldsymbol{\upsilon}_j\cdot\nabla)\boldsymbol{\upsilon}_j] = n_jq_j(\mathbf{E} + \boldsymbol{\upsilon}_j\times\mathbf{B}) - \nabla p_j - \overline{\nu_{jk}}n_jm_j(\boldsymbol{\upsilon}_j - \boldsymbol{\upsilon}_k)$$

3. Energy/Equation of State:

$$p_jn_j^{-\gamma} = const.$$

Maxwell's Equations

$$\nabla\cdot\mathbf{B} = 0 \qquad\qquad \nabla\cdot\mathbf{E} = \rho/\epsilon_0$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

with

$$\rho = q_e n_e + q_i n_i = e(-n_e + Zn_i)$$

$$\mathbf{j} = q_e n_e \mathbf{\upsilon}_e + q_i n_i \mathbf{\upsilon}_i = e(-n_e \mathbf{\upsilon}_e + Zn_i \mathbf{\upsilon}_i)$$

$$= -en_e(\mathbf{\upsilon}_e - \mathbf{\upsilon}_i) \qquad \text{(quasineutral)}$$

Accounting

Unknows		Equations	
n_e, n	2	Continity e,i	2
$\mathbf{\upsilon}_e, \mathbf{\upsilon}_i$	6	Momentum e,i	6
p_e, p_i	2	State e,i	2
E, **B**	6	Maxwell	8
	16		18

But 2 of Maxwell ($\nabla \cdot$ equs) are redundant because can be deduced from others: e.g.

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B})$$

and

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0 = \mu_0 \nabla \cdot \mathbf{j} + \frac{1}{c^2}\frac{\partial(\nabla \cdot \mathbf{E})}{\partial t}$$

so 16 equs for 16 unknowns.

Equations still very difficult and complicated mostly because it is Nonlinear.
In some cases can get a tractable problem by 'linearizing'. That means, take some known equilibrium solution and suppose the deriation (pertubation) from it is small so we can retain only the 1st linear terms and not the others.

4.5 One-fluid (Magnetohydrodynamic or MHD) Equations

Reduction of Fluid Approach to the Single Fluid Equations

So far we have been using fluid equations which apply to electrons and ions separately. These are called 'two fluid' equations because we always have to keep track of both fluids separately.

A further simplification is possible and useful sometimes by combining the electron and ion equations together to obtain equations governing the plasma viewed as a 'single fluid'. The MHD model is applicable only when charge separation is negligible. The condition for it is that the length scales should be larger than the Debye length and the time scales larger than the inverse of plasma frequency. When we consider non-relativistic and slowly varying motions of plasmas under the action of mechanical and magnetic forces, the MHD model is the appropriate model to apply. We should keep in mind that the main limitation of the MHD model is that it cannot be applied to high-frequency phenomena which may involve charge separation (plasma oscillations or electromagnetic waves in plasmas).

Recall 2-fluid equations

Continuity (C_j)

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0$$

Momentum (M_j)

$$m_j n_j [\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j] = n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla p_j + \mathbf{F}_{jk}$$

$$\text{where } \mathbf{F}_{jk} = -\overline{v_{jk}} n_j m_j (\mathbf{v}_j - \mathbf{v}_k)$$

Now we rearrange these 4 equations (2×2 species) by adding and subtracting appropriately to get new equations governing the new variables:

Mass density:

$$\rho_m = n_e m_e + n_i m_i$$

C of M velocity:

$$\mathbf{V} = (n_e m_e v_e + n_i m_i v_i)/\rho_m$$

Charge density:

$$\rho_q = q_e n_e + q_i n_i$$

Electric Current density

$$\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i = q_e n_e (\mathbf{v}_e - \mathbf{v}_i)$$

Total Pressure

$$p = p_e + p_i$$

1st equation: take $m_e \times C_e + m_i \times C_i \rightarrow$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0 \qquad \text{Mass Conservation}$$

2nd take $q_e \times C_e + q_i \times C_i \rightarrow$

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{j} = 0 \qquad \text{Charge Conservation}$$

3rd take $M_e + M_i$ This is a bit more difficult. RHS becomes:

$$\sum n_j q_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla p_j + \mathbf{F}_{jk} = \rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla(p_e + p_i)$$

(we use the fact that $\mathbf{F}_{ei} = -\mathbf{F}_{ie}$ so no net friction).

L.H.S. is

$$\sum_j m_j n_j (\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla) v_j$$

The difficulty here is that the convective term is no-linear and so does not easily lend itself to reexpression in terms of the new variables. But note that since $m_e << m_i$ the contribution from electron momentum is usually much less than that from ions. So we ignore it in this equation. To the same degree of approimation $\mathbf{V} \simeq \mathbf{v}_i$: The CM velocity is the ion velocity. Thus for the LHS of this momentum equation we take

$$\sum_j m_j n_j (\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla) v_j \simeq \rho_m (\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla) \mathbf{V}$$

so:

$$\rho_m\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} = \rho_q\mathbf{E} + \mathbf{j}\times\mathbf{B} - \nabla p$$

Finally we take $\frac{q_e}{m_e}M_e + \frac{q_i}{m_i}M_i$ to get:

$$\sum_j n_j q_j\left[\frac{\partial}{\partial t} + \boldsymbol{\upsilon}_j\cdot\nabla\right]\boldsymbol{\upsilon}_j = \sum_j\left\{\frac{n_j q_j^2}{m_j}(\mathbf{E} + \boldsymbol{\upsilon}_j\times\mathbf{B}) - \frac{q_j}{m_j}\nabla p_j + \frac{q_j}{m_j}\mathbf{F}_{jk}\right.$$

Again several difficulties arise which it is not very profitable to deal with rigorously. Observe that the LHS can be written (using quasineutrality) as $\rho_m\frac{\partial}{\partial t}(\mathbf{j}/\rho_m)$ (ignoring m_e/m_i term) provided we discard the term in $(\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}$. (Think of this as a linearization of this equation) [The $(\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}$ convective term is a term which is not satisfactorily dealt with in the single fluid equations.]

In the R.H.S. we use quasineutrality again to write

$$\sum_j\frac{n_j q_j^2}{m_j}\mathbf{E} = n_e^2 q_e^2\left(\frac{1}{n_e m_e} + \frac{1}{n_i m_i}\right)\mathbf{E} = n_e^2 q_e^2\frac{m_i n_i + m_e n_e}{n_e m_e n_i m_i}\mathbf{E}$$

$$= -\frac{q_e q_i}{m_e m_i}\rho_m\mathbf{E}$$

$$\sum_j\frac{n_j q_j^2}{m_j}\boldsymbol{\upsilon}_j = \frac{n_e q_e^2}{m_e}\boldsymbol{\upsilon}_e + \frac{n_i q_i^2}{m_i}\boldsymbol{\upsilon}_i \qquad [n_i q_i = -n_e q_e]$$

$$= \frac{q_e q_i}{m_e m_i}\left\{\frac{n_e q_e m_i}{q_i}\boldsymbol{\upsilon}_e + \frac{n_i q_i m_e}{q_e}\boldsymbol{\upsilon}_i\right\}$$

$$= -\frac{q_e q_i}{m_e m_i}\{n_i m_i\boldsymbol{\upsilon}_e + n_e m_i\boldsymbol{\upsilon}_i\}$$

$$= -\frac{q_e q_i}{m_e m_i}\{n_e m_e\boldsymbol{\upsilon}_e + n_i m_i\boldsymbol{\upsilon}_i -$$

$$\left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)(q_e n_e\boldsymbol{\upsilon}_e + q_i n_i\boldsymbol{\upsilon}_i\}$$

$$= -\frac{q_e q_i}{m_e m_i}\left\{\rho_m\mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)\mathbf{j}\right\}$$

Also, remembering,

$$\mathbf{F}_{ei} = -\overline{\nu_{ei}}n_e m_e(\boldsymbol{\upsilon}_e - \boldsymbol{\upsilon}_i) = -\mathbf{F}_{ie}$$

$$\sum_j\frac{q_j}{m_j}\mathbf{F}_{jk} = -\overline{\nu_{ei}}\left(n_e q_e - n_e q_i\frac{m_e}{m_i}\right)(\boldsymbol{\upsilon}_e - \boldsymbol{\upsilon}_i)$$

$$= -\overline{\nu_{ei}}\left(1 - \frac{q_e m_e}{q_i m_i}\right)\mathbf{j}$$

So we get

$$\rho_m\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{\rho_m}\right) = \frac{q_e q_i}{m_e m_i}\left[\rho_m\mathbf{E} + \left\{\rho_m\mathbf{V} - \left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)\mathbf{j}\right\}\times\mathbf{B}\right.$$

$$- \frac{q_e}{m_e}\nabla p_e - \frac{q_i}{m_i}\nabla p_i - \left(1 - \frac{q_e m_e}{q_i m_i}\right)\overline{\nu_{ei}}\mathbf{j}$$

Regroup after multiplying by $\frac{m_e m_i}{q_e q_i \rho_m}$:

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = -\frac{m_e m_i}{q_e q_i}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{\rho_m}\right) + \frac{1}{\rho_m}\left(\frac{m_i}{q_i} + \frac{m_e}{q_e}\right)\mathbf{j}\times\mathbf{B}$$

$$- \left(\frac{q_e}{m_e}\nabla p_e + \frac{q_i}{m_i}\nabla p_i\right)\frac{m_e m_i}{q_e q_i\rho_m}$$

$$- \left(1 - \frac{q_e m_e}{q_i m_i}\right)\frac{m_e m_i}{q_e q_i\rho_m}\overline{\nu_{ei}}\mathbf{j}$$

Notice that this is an equation relating the Electric field in the frame moving with the fluid (L.H.S) to things depending on current, J. i.e. this is a generalised form of Ohm's Law.

One essentially never deals with this full generalised Ohm's law. Make some approximations recognizing the physical significance of the various R.H. S. terms.

$\frac{m_e m_i}{q_e q_i} \frac{\partial}{\partial t} (\frac{\mathbf{j}}{\rho_m})$ arises from electron inertia. it will be negligible for low enough frequency.

$\frac{1}{\rho_m} (\frac{m_i}{q_i} + \frac{m_e}{q_e})\mathbf{j} \times \mathbf{B}$ is called the Hall Term

and arises because current flow in a B-filed tends to be diverted across the magnetic field. It is also often dropped but the justification for doing so is less obvious physically.

$\frac{q_i}{m_i} \nabla p_i$ term << $\frac{q_e}{m_e} \nabla p_e$ for comparable pressures, and the later is ~the Hall term.

Last term in **J** has a coefficient, ignoring m_e/m_i c.f. 1 which is

$$\frac{m_e m_i \overline{\nu_{ei}}}{q_e q_i (n_i m_i)} = \frac{m_{ei} \overline{\nu_{ei}}}{q_e^2 n_e} = \eta \quad \text{the resistivity}$$

Hence dropping electron inertia, Hall term $\propto$ pressure the Ohm's law becames:

$$\mathbf{E} + \boldsymbol{\upsilon} \times \mathbf{B} = \eta \mathbf{j}$$

Final equation needed: state:

$$p_e n_e^{-\gamma_e} + p_i n_i^{-\gamma_i} = const.$$

Take quasi-neutrality$\Rightarrow$ $n_e \propto n_i \propto \rho_m$

Take $\gamma_e = \gamma_i$:

$$p \rho_m^{-\gamma} = const.$$

4.6 Summary of Single Fluid Equations: MHD

Mass Conservation:

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0$$

Charge Conservation:

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

Momentum:

$$\rho_m (\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla)\mathbf{V} = \rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p$$

Ohm's Law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

Eq. of State:

$$p \rho_m^{-\gamma} = const.$$

Heuristic Derivation/Explanation
Mass/Charge: obvious
Momentum:

$$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B} - \nabla p$$

$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V}$: Rate of change of fluid total momentum density
$\rho_q \mathbf{E}$: Body force density exerted on fluid's charge
$\mathbf{j} \times \mathbf{B}$: mag. force on its current
∇p : Pressure

Ohm's Law
The electric field 'seen' by a moving (conducting) fluid is $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{E}_v$ electric field in frame in which fluid is at rest. This is equal to 'resistive' electric field $\eta \mathbf{j}$:
$\mathbf{E}_v = \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$
The $\rho_q \mathbf{E}$ term is generally dropped because it is much smaller than the $\mathbf{j} \times \mathbf{B}$ term. To see this take orders of magnitude:

$$\nabla \cdot \mathbf{E} = \rho_q / \epsilon_0 \Rightarrow \rho_q \sim E \epsilon_0 / L$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \left(+ \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) \Rightarrow j (= \sigma E) \sim \frac{B}{\mu_0 L}$$

so

$$\rho_q E \sim \frac{E^2 \epsilon_0}{L} \sim \frac{\epsilon_0}{L} \left(\frac{B}{\mu_0 \sigma L} \right)^2$$

$$\frac{\rho_q E}{jB} \sim \frac{\epsilon_0}{L} \left(\frac{B}{\mu_0 \sigma L} \right)^2 \frac{\mu_0 L}{B^2} \sim \frac{1}{c^2} \frac{1}{(\mu_0 \sigma L)^2}$$

Take a typical vale for σ for even a small cold plasma, say 1 eV, 1 cm; $\sigma \simeq 2 \times 10^3 \Omega^{-1} m^{-1}$ then

$$\begin{aligned} \frac{1}{c^2} \frac{1}{(\mu_0 \sigma L)^2} &= \frac{1}{(3 \times 10^8 \times 4\pi \times 10^{-7} \times 2 \times 10^3 \times 10^{-2})^2} \\ &\simeq 10^{-8} \end{aligned}$$

Conclusion: $\rho_q E$ force is much smaller than $\mathbf{j} \times \mathbf{B}$ for essentially all practical cases. Ignore it.
Nomally, also, one uses MHD only for low frequency phenomena so the Maxwell displacement current $\frac{1}{c^2} \frac{\partial E}{\partial t}$ can be ignored.
Also, we shall not need Poisson's equation because that is taken care of by quasi-neutrality.