MHD equilibrium and instabilities

5.1 Introduction

Magnetohydrodynamics is a branch of plasma physics dealing with dc or low frequency effects in fully ionized magnetized plasma. In this chapter we will study both "ideal" (i.e. zero resistance) and resistive plasma, introducing the concepts of stability and equilibrium, "frozen-in" magnetic fields and resistive diffusion. We will then describe the range of low-frequency plasma waves supported by the single fluid equations.

5.2 Ideal MHD – Force Balance

The ideal MHD equations are obtained by setting resistivity to zero:

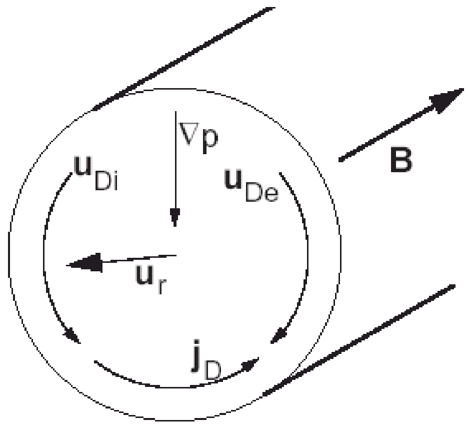
$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p$$

In equilibrium, the time derivative is ignored and the force balance equation:

$$\mathbf{j} \times \mathbf{B} = \nabla p$$

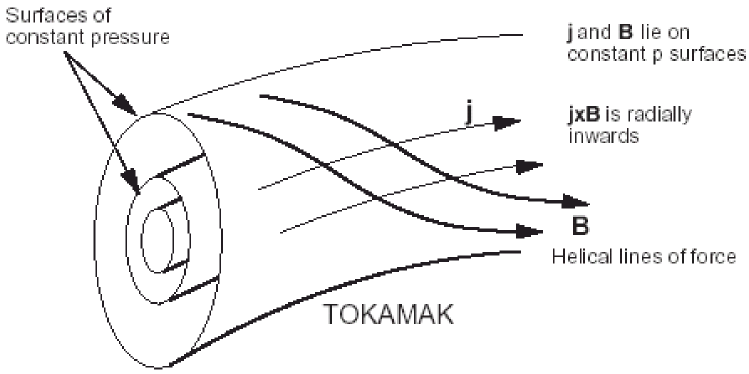
describes the balance between plasma pressure force and Lorentz forces. Such a balance is shown schematically in Fig. 6.1.



What are the consequences of this simple expression? The current required to balance the plasma pressure is found by taking the cross product with **B**. Then

$$\mathbf{j}_{\perp} = \frac{\mathbf{B} \times \nabla p}{B^2} = (kT_e + kT_i) \frac{\mathbf{B} \times \nabla n}{B^2}$$

This is simply the diamagnetic current! The force balance says that j and B are perpendicular to ∇p. In other words, j and B must lie on surfaces of constant pressure. This situation is shown for a tokamak in Fig. 6.2. The current density and magnetic lines of force are constrained to lie in surfaces of constant plasma pressure. The angle between the current and field increases as the plama pressure increases.



In the direction parallel to B:

$$\frac{\partial p}{\partial z} = 0$$

Plasma pressure is constant along a field line. (Why is this?)

5.2.1 Magnetic pressure

From Maxwell's equation

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

$$\nabla p = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0} [(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2} \nabla B^2]$$

or

$$\nabla(p + \frac{B^2}{2\mu_0}) = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla)\mathbf{B}$$

If the magnetic field lines are straight and parallel, so that the right side vanishes. Pressure balance gives

$$p + \frac{B^2}{2\mu_0} = 0$$

and $\frac{B^2}{2\mu_0}$ represents the magnetic pressure. Thus, if the plasma exhibits a pressure gardient, there is a corresponding gradient in the magnetic pressure that ensures the total pressure is constant in the plasma fluid.

Note the ratio of particle pressure to magnetic pressure is β,

$$\beta = \frac{nkT}{B^2/2\mu_0}$$

In most fusion plasmas at low pressure $\beta << 1$. For tokamaks, β does not exceed a few percent.

Example Pressure-balanced plasma column: θ –pinch

So called because plasma currents flow in θ-direction.

Take the column to be ∞ length, uniform in z-dir. By symmetry, i

B has only z-component
j has only θ component
∇p has only r component

So we only need
force

$$(\mathbf{j} \times \mathbf{B})_r - (\nabla p)_r = 0$$

Ampere

$$(\nabla \times B)_\theta = (\mu_0 \mathbf{j})_\theta$$

i.e.

$$j_\theta B_z - \frac{\partial}{\partial r} p = 0$$

$$- \frac{\partial}{\partial r} B_z = \mu_0 j_\theta$$

Eliminate j:

$$- \frac{B_z}{\mu_0} \frac{\partial B_z}{\partial r} - \frac{\partial p}{\partial r} = 0$$

i.e.

$$\frac{\partial}{\partial r} (\frac{B_z^2}{2\mu_0} + p) = 0$$

Solution

$$\frac{B_z^2}{2\mu_0} + p = const$$

$$\frac{B_z^2}{2\mu_0} + p = \frac{B_{zext}^2}{2\mu_0}$$

Think of this as a pressure equation. Equlbria says total pressure = const.

$$\underbrace{\frac{B_z^2}{2\mu_0}} + \underbrace{p} = const$$

Magnetic pressure Kinetic
pressure

Low β equilibria: Force-Free plasmas.

In the cases the ratio of kinetic to magnetic pressure is small, $\beta << 1$, and we can approximately ignore ∇p. Such as equilibrium is called 'force free'.

$$\mathbf{j} \times \mathbf{B} = 0$$

implies **j** and **B** are parallel.
i.e.

$$\mathbf{j} = \mu(r)\mathbf{B}$$

Where μ is a scalar, which in principle can be a function of space.
Current flows along field lines, but do not across.
Take divergence:

$$0 = \nabla \cdot \mathbf{j} = \nabla \cdot (\mu(r)\mathbf{B}) = \mu\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla\mu$$
$$= \mathbf{B} \cdot \nabla\mu$$

This equation means that μ cannot vary along a magnetic field line. In general, μ can have different values on different field lines, but it has to be a constant on one field line.

The simplest case is to consider μ to be a costant. – linear force-free field

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j} = \mu_0\mu\mathbf{B}$$

This is a somewhat more convenient form because it is linear in **B** (for specified μ).

A linear equation can in general be solved by a series expansion. SInce it is still a vector equation rather than a scalar equation, obtaining a general solution by series expansion is slightly complicated. Here we shall not discuss this general solution, but only consider the solution with cylindrical symmetry.

Written in cylindrical coordinates assuming cylindrical symmetry (i.e. no variation of any quantity in θ or z directions):

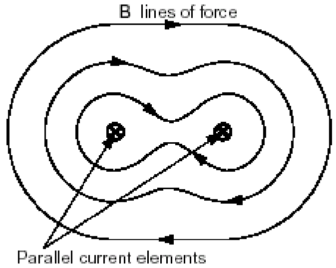
$$-\frac{dB_z}{dr} = \mu B_\theta$$
$$\frac{1}{r}\frac{d}{dr}(rB_\theta) = \mu B_z$$

leads to a Bessel function solution

$$B_z = B_0 J_0(\mu r)$$
$$B_\theta = B_0 J_1(\mu r)$$

6.2.2 Magnetic tension

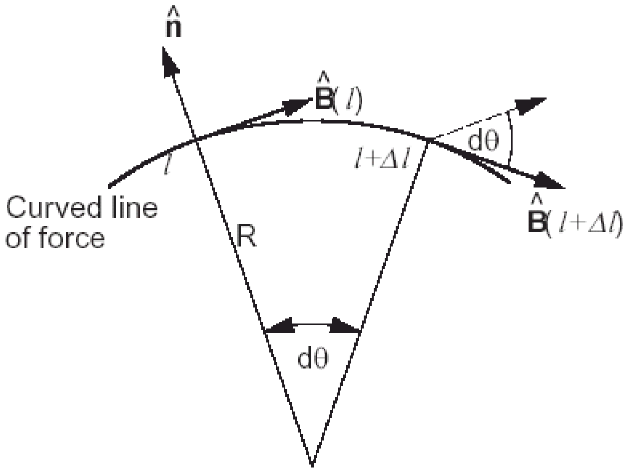
It is known that two wires carrying parallel currents attract as if the magnetic lines of force were under tension.



Magnetic tension is described by the term

$$(\mathbf{B} \cdot \nabla)\mathbf{B} = B_x \frac{\partial}{\partial x}(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) + \ldots$$

If the magnetic lines of force are straight and parallel then B = $B_x\mathbf{i}$ and $(\mathbf{B}\cdot\nabla)\mathbf{B} = 0$. This term is only important if the magnetic lines of force are curved.



To show this, consider the geometric construction as shown and let $\hat{\mathbf{B}} = \mathbf{B}/|\mathbf{B}|$ be the unit vector in the direction of the field. By definition,

$$\hat{\mathbf{B}} \cdot \nabla\hat{\mathbf{B}} = \frac{\partial \hat{\mathbf{B}}}{\partial l}$$

where l is the coordinate along the line of force.
In terms of infinitesimal quantities, we have

$$\frac{\Delta\hat{\mathbf{B}}}{\Delta l} = \frac{\hat{\mathbf{B}}(l + \Delta l) - \hat{\mathbf{B}}(l)}{\Delta l}$$

It is clear that

$$\hat{\mathbf{B}}(l + \Delta l) - \hat{\mathbf{B}}(l) = -\hat{\mathbf{n}}d\theta$$

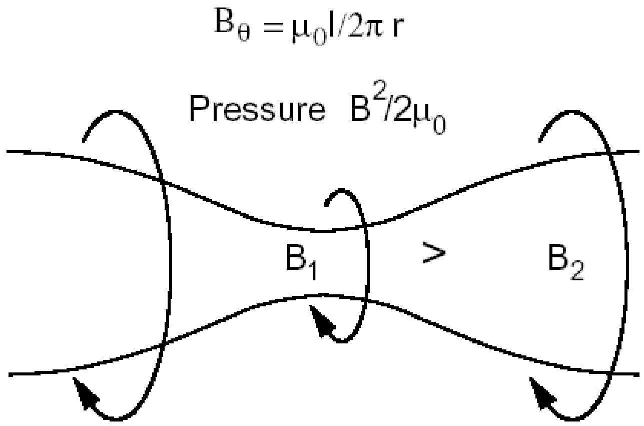
Where $\hat{\mathbf{n}}$ is the unit normal to the field line, while $\Delta l = Rd\theta$, therefore

$$\hat{\mathbf{B}} \cdot \nabla\hat{\mathbf{B}} = -\frac{\hat{\mathbf{n}}}{R}$$

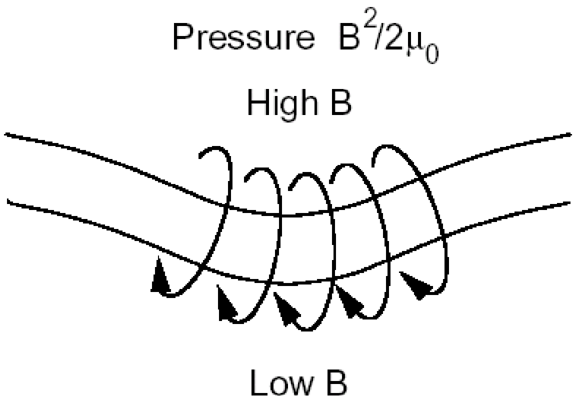
so that the magnetic tension in inversely poportional to the radius of curvature of the magnetic field line. When the lines of force are curved, they behave as though under a tension force B^2/μ_0 per unit area. As show above this produces a force that is perpendicular to **B** (parallel to $\hat{\mathbf{n}}$) and inversely proportional to the radius of curvature. Thus the lines of force can be regarded as elastic cords under tension B^2/μ_0.

6.2.3 Stability of plasma columns

Consider an unmagnetized linear plasma pinch carrying current I as shown.



We have $B_z = 0$ and $B_\theta = \mu_0 I/(2\pi r)$. If the radius of the plasma channel in some region gets smaller, the magnetic field gets larger. This increases the magnetic pressure with the result that the channel shrinks further. Though the plasma pressure increases also, it spreads out along the column and cannot balance the locally high magnetic pressure. This is an unstable equilibrium and the instability is known as the m = 0 sausage instability.



If the column develops a kink, the increased pressure and tension on the high B side increases the bending. This is known as the m = 1 kink instability. Instabilities can be prevented by adding a longitudinal magnetic field B_z to "stiffen" the plasma. The $B_z^2/2\mu_0$ pressure resists the m = 0 mode and the tension B_z^2/μ_0 counters the bending.

5.3 Frozen-in Magnetic Fields

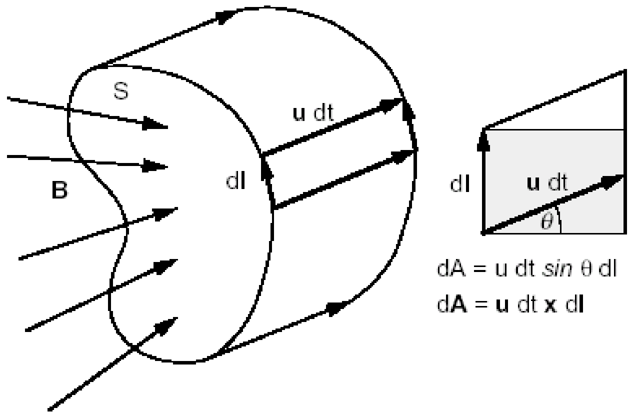
We examine the behaviour of the plasma when the magnetic field changes with time. The latter generates an emf given by Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Taking the curl of the infinte conductivity Ohm's law and using Faraday's law then gives

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

This relation will allow us to study the behaviour of magnetic flux through a surface moving with the plasma at velocity u.



With reference to the Fig, the magnetic flux through the surface S is

$$\Phi_s = \int \mathbf{B} \cdot d\mathbf{s}$$

This flux changes either through time variations in B or due to the surface moving with the plasma fluid element to a new position where B is different — the convective term. The first contribution is just

$$\frac{\partial \Phi_s}{\partial t} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

To evaluate the convective component, let dl be a unit vector lying on the circumference of S. Due to the motion of the plasma fluid element, this element sweeps out an area $d\mathbf{A} = \mathbf{u}dt \times d\mathbf{l}$ in time dt. The associated change in magnetic flux through this area element is

$d\Phi = \mathbf{B} \cdot d\mathbf{A}$. The total rate of change of flux is found by integrating along around the circumference of S:

$$\begin{aligned}\frac{d\Phi}{dt} &= \oint \mathbf{B} \cdot (\mathbf{u} \times d\mathbf{l}) \\ &= \oint (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{l} \\ &= \int \nabla \times (\mathbf{B} \times \mathbf{u}) \cdot d\mathbf{S}\end{aligned}$$

where we have applied Stokes' theorem in the final step. If now we add the two components of the changing flux we obtain

$$\begin{aligned}\frac{D\Phi_s}{Dt} &= \int [\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{B} \times \mathbf{u})] \cdot d\mathbf{S} \\ &= 0\end{aligned}$$

The magnetic flux through S doesn't change with time. This implies that the magnetic lines of force are "frozen" into the plasma and are transported with it. Thus, if the plasma expands, the field strength decreases (reminiscent of magnetic moment). If the flux did change, there would be an induced emf established to oppose the change (Lenz's law). This would establish an E/B drift that would change the plasma shape to preserve the magnetic flux. If the plasma is resistive, currents can flow which can dissipate the magnetic energy as we shall soon see.

5.4 Resistive Diffusion

Ideal MHD assumes the plasma is a perfect conductor. Let us investigate the relationship between plasma and magnetic pressure when finite plasma resistivity allows the magnetic field to diffuse into the conducting fluid. The Ohm's law is now

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j}$$

and Faraday's law gives

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times (\eta \mathbf{j} - \mathbf{u} \times \mathbf{B}) \\ &= -\nabla \times (\frac{\eta}{\mu_0} \nabla \times \mathbf{B} - \mathbf{u} \times \mathbf{B}) \\ &= -\frac{\eta}{\mu_0} \nabla \times \nabla \times \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}) \end{aligned}$$

with the vector identity

$$\nabla \times \nabla \times \mathbf{B} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \boldsymbol{\nabla}^2 B = -\boldsymbol{\nabla}^2 B$$

we finally obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \boldsymbol{\nabla}^2 B + \nabla \times (\mathbf{u} \times \mathbf{B})$$

The is the finite resistivity version of the $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$ for ideal fluid. The right side now includes a diffusive term that accounts for resistive decay of the magnetic field energy. We make some order of magnitude comparisons of the two terms:

$$\nabla \times (\mathbf{u} \times \mathbf{B}) \sim \frac{\Delta(uB)}{\Delta x} \sim \frac{uB}{L}$$

where L is the scale length for change of u or B by a factor of two or three. For the diffusive term we have

$$\frac{\eta}{\mu_0} \boldsymbol{\nabla}^2 B \sim \frac{\eta}{\mu_0} \frac{\Delta(\partial B/\partial x)}{\Delta x} \sim \frac{\eta}{\mu_0} \frac{B}{L^2}$$

We now define the magnetic Reynold's number R_m as the ratio of the estimates of the convective and diffusive terms:

$$R_m = \frac{uB}{L} / (\frac{\eta}{\mu_0} \frac{B}{L^2}) = \frac{\mu_0 u L}{\eta}$$

For low resistivity plasma ($\eta \to 0$), $R_m >> 1$ and the above equation reduces to $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$ describing frozen-in flow. For higher resistivity, $R_m << 1$ the plasma behaves like an ordinary conductor ($\mathbf{u} \times \mathbf{B} = \mathbf{0}$) and we obtain a diffusion equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \boldsymbol{\nabla}^2 B$$

which can be solved using separation of variables. As a rough estimate, we write

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \frac{\mathbf{B}}{L^2}$$

giving

$$\mathbf{B} = \mathbf{B} \exp(-t/\tau_R)$$

where

$$\tau_R = \frac{\mu_0 L^2}{\eta}$$

is the characteristic time for magnetic penetration into the conducting fluid. In the case where B diffuses through a conducting plasma (u = 0), the moving B lines of force induce eddy currents that ohmically heat the plasma. This energy comes from the magnetic field. The induced current is $\mu_0 \mathbf{j} = \nabla \times \mathbf{B} \sim B/L$ and the energy lost per unit volume by the field to heat the plasma in time τ_R is

$$\eta j^2 \tau_R = \eta\left(\frac{B}{\mu_0 L}\right)^2\left(\frac{\mu_0 L^2}{\eta}\right) = \frac{B^2}{\mu_0}$$

This corresponds nicely with the initial magnetic field energy density. Thus τ_R is the time for the magnetic energy to be resistively dissipated into heat.

Plasma Instabilities

The MHD configurations and flow phenomena we have studied so far assume that space plasma systems are in stable equlibrium. However, evidence shows that plasma systems are often unstable. We introduce the basic concepts and methods to discuss how steay-state equilibrium structures become unstable, we will limit dicussion in this chapter to MHD instablilites only primarily associated with large-scale phenomena.

6.1 Classification of Instabilities

Instabilities may be classified accoding to the type of free energy available to drive them. There are four categories:

1. Streaming instabilities.

In this case, either a beam of energetic particles travels through the plasma, or a current is driven through the plasma so that the different species have drifts relative to one another. The drift energy is used to excite waves, and oscillation energy is gained at the expense of the drift energy in the unperturbed state.

2. Rayleigh-Taylor instabilities.

In this case, the plasma has a density gradient or a sharp boundary, so that it is not uniform. In addition, an external, nonelectromagnetic force is applied to the plasma. It is this force which drives the instability.

3. Universal instablities.

Even when there are no obvious driving forces such as an electric or a gravitational field, a plasma is not in perfect thermodynamic equilbrium as long as it is confined. The plasma pressure tends to make the plasma expand, and the expansion energy can drive an instability. This type of free energy is always present in any finite plasma, and the resulting waves are called universal instabilities.

4. Kinetic instabilities.

In fluid theory the velocity distributions are assumed to be Maxwellian. If the distributions are in fact not Maxwellian, there is a deviation from thermodynamic equilibrium; and instabilities can be derived by the anisotropy of the velocity distribution. For instance, if $T_{//}$ and $T_{\perp}$ are different, an instability called the modified Harris instability can arise. In mirror devices, there is a deficit of particles with large $v_{//}/v_{\perp}$ because of the loss cone; this anisotropy gives rise to a "loss core instability".

The instabilities driven by anisotropy cannot be described by fluid theory and a detailed treatment of them is beyond the scope of this course.

6.2 Methods of Instability Analysis

A method used to study equilibrium problems imagines the system to undergo a small displacement as the result of the application of an arbitray force. If the force increases the displacement and thereby deforms the system, the system is said to be unstable. If, however, the effects of the force are damped and the system returns to the initial configuration, the system is considered stable.

1. Normal mode: examine whether the pertubation grows or damps by studying the motion of the particles in the immediate neighborehood.
2. energy principle methods: by calculating the energy of the initial and final states.

Consider two point masses in a one-dimensional potental field V(x) as shown.
A small pertubation applied to point A will cause the mass to oscillate about the equilibrium point while perturbation applied to at point B will accelerate the mass away from the equilibrium point. System A is stable and system B is unstable.

Let the coordinate of the equilbrium position be given by x_0 and the force $F(x)$. The equation of motion of the mass m at position x, obtained by Taylor expansion about the point x_0 :

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F(x) \\ &= F(x_0) + F'(x_0)(x - x_0) + \ldots \\ &= F'(x_0)(x - x_0) + \ldots \end{aligned}$$

where x_0 is the equilibrium position, $F(x_0) = 0$
let $\xi = x - x_0$ and ignored higher-order terms.

$$m\frac{d^2\xi}{dt^2} = F'(x_0)\xi$$

The solution of this equation:

$$\xi = \xi_0 \exp[\frac{F'(x_0)}{m}]^{1/2} t$$
$$= \xi_0 \exp(i\omega t)$$

where $\omega^2 = -F'(x_0)/m$.
A: $F'(x_0) < 0, \omega$ real, the solution is oscillatory
B: $F'(x_0) > 0$, the distubance grows exponentially.

As another example, condider two vessels in which thers are two kinds of fluids in a gravitaonal filed. Let the fluid on the top in one case (left) be lighter than the one on the bottom, and let the reverse be true in the other case (right). Both systems are initally in equilibrium. Introduce now a small pertubation in the form of waves, to the interfaces of two fluids.
Left: the waves will oscillate about the equilibrium and will eventually damp out
Right: the waves will grow, which will ead to the interchange of the positions of the upper and lower fluids.
The lower-energy state is reached by lowering the potential energy.

6.3 Two-stream Instability (Buneman instability)

As a simple example of a steaming instability, consider a uniform plasma in which the ions are stationary and the electrons have a velocity υ_0 relative to the ions. Let the plasma be cold ($kT_e = kT_i = 0$), and let there be no magnetic field ($B_0 = 0$). We first separate teh dependent variables into two parts: an "equilibrium" part indicated by a subscript 0, and a "pertubation" part indicated by a subscript 1.

$$n_e = n_0 + n_1, \boldsymbol{\upsilon}_e = \boldsymbol{\upsilon}_0 + \boldsymbol{\upsilon}_1, \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

The equations of motion for the ions and the electrons are, to first order,

$$Mn_0 \frac{\partial \boldsymbol{\upsilon}_{i1}}{\partial t} = en_0 \mathbf{E}_1$$
$$mn_0[\frac{\partial \boldsymbol{\upsilon}_{e1}}{\partial t} + (\upsilon_0 \cdot \nabla)\boldsymbol{\upsilon}_{e1}] = -en_0 \mathbf{E}_1$$

We look for electrostic waves of the form

$$\mathbf{E}_1 = Ee^{i(kx-\omega t)} \hat{x}$$

where $\hat{x}$ is the direction of υ_0.

$$-i\omega Mn_0 \boldsymbol{\upsilon}_{i1} = en_0 \mathbf{E}_1 \Rightarrow \boldsymbol{\upsilon}_{i1} = \frac{ie}{M\omega} E\hat{x}$$
$$mn_0(-i\omega + ik\upsilon_0)\boldsymbol{\upsilon}_{e1} = -en_0 \mathbf{E}_1 \Rightarrow \boldsymbol{\upsilon}_{e1} = -\frac{ie}{m} \frac{E\hat{x}}{\omega - k\upsilon_0}$$

The velocities $\boldsymbol{\upsilon}_{j1}$ are in the x direction, and we may omit the subscript x. The ion equation of continuity yields

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla \cdot \boldsymbol{\upsilon}_{i1} = 0 \Rightarrow n_{i1} = \frac{k}{\omega} n_0 \upsilon_{i1} = \frac{ien_0 k}{M\omega^2} E$$

Note that the orther terms in $\nabla \cdot (n\boldsymbol{\upsilon}_i)$ vanish because $\nabla n_0 = \boldsymbol{\upsilon}_{0i} = 0$. The electron equation of continuity is:

$$\frac{\partial n_{i1}}{\partial t} + n_0\nabla \cdot \mathbf{v}_{e1} + (\mathbf{v}_0 \cdot \nabla)n_{e1} = 0$$

$$(-i\omega + ikv_0)n_{e1} + ikn_0v_{e1} = 0$$

$$n_{e1} = \frac{kn_0}{\omega - kv_0}v_{e1} = -\frac{iekn_0}{m(\omega - kv_0)^2}E$$

Since the unstable waves are high-frequency plasma oscillations, we may not use the plasma approximation but must use Poisson's equation:

$$\epsilon_0\nabla \cdot \mathbf{E}_1 = e(n_{i1} - n_{e1})$$

$$ik\epsilon_0E = e(ien_0kE)[\frac{1}{M\omega^2} + \frac{1}{(\omega - kv_0)^2}$$

The dispersion relation is found upon dividing by $ik\epsilon_0E$:

$$1 = \omega_p^2[\frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2}]$$

Let us see if oscillations with real k are stable or unstable. If all the roots ω_j are real, each root would indicate a possible oscillation

$$\mathbf{E}_1 = Ee^{i(kx-\omega_jt)}\hat{x}$$

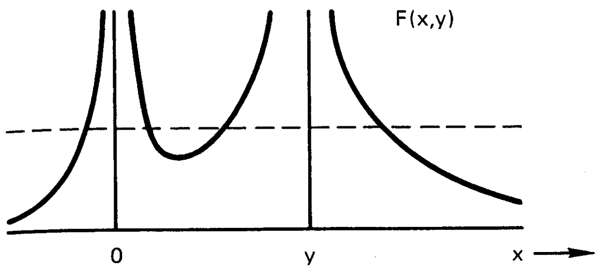
If some of the roots are complex, they will occur in complex conjugate pairs. Let these complex roots be written

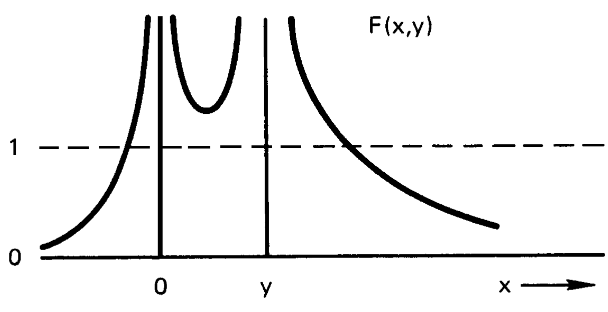
$$\omega_j = \alpha_j + i\gamma_j$$

The time dependence is now given by

$$\mathbf{E}_1 = Ee^{i(kx-\alpha_jt)}e^{\gamma_jt}\hat{x}$$

Positive Im(ω) indicates an exponentially growing wave; negative Im(ω) indicates a damped wave. Since the roots ω_j occur in conjugate paris, one of these will always be unstalbe unless all the roots are real. The damped roots are not self-excited and are not of interesting.





The dispersion relation can be analyzed without actually solving the fouth-order equation. Let us define

$$x \equiv \frac{\omega}{\omega_p}; y = \frac{kv_0}{\omega_p}$$

$$1 = \frac{m/M}{x^2} + \frac{1}{(x - y)^2} = F(x,y)$$

For any give value of y, we can plot $F(x,y)$ as a funciton of x. THis function will have singularites at x=0 and x=y. The intersections of this curve with the line $F(x,y) = 1$ gives the values of x satisfying the dispersion relation. However, we are only guaranteed two real roots. The other two can be complex. This occurs, for example, when the local minimum value of $F(x,y) > 1$. These complex roots correspond to unstable waves.
The maximum growth rate predicted is, for $m/M << 1$,

$$\mathrm{Im}(\frac{\omega}{\omega_p}) \simeq (\frac{m}{M})^{1/3}$$

6.4 The "Gravitational" Instability (Rayleigh-Taylor Instability)

The famous Rayleigh-Taylor configuration corresponds to the case in which **F** is the gravitational force. A magnetized fluid exists on one side and a magnetic field on the other sider. Let the plane boundary be in teh yz-plane and let thre be a density gradient in the -x-direction. Let $\mathbf{B}_0$ be in the z-direction. We assume the plasma β is low so that we can let $kT_e = kT_i = 0$. This implies that there is no diamagnetic current (due to ∇n).

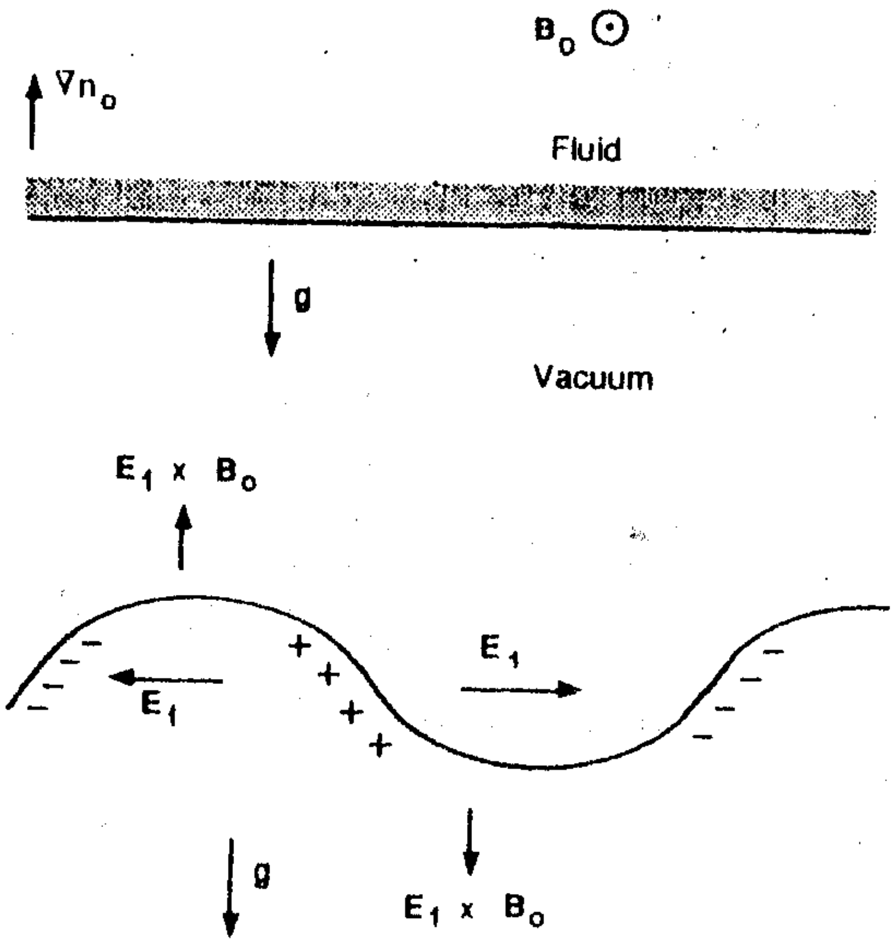
In the equilbrium state, the ions obey the equation

$$Mn_0(v_0 \cdot \nabla)v_0 = qn\mathbf{v}_0 \times \mathbf{B}_0 + Mn_0\mathbf{g}$$

if g is a constant, v_0 will be also; and $(v_0 \cdot \nabla)v_0$ vanishes. Taking the cross product of the above equation with $\mathbf{B}_0$, we find

$$\mathbf{v}_0 = \frac{M}{q}\frac{\mathbf{g} \times \mathbf{B}_0}{B_0^2} = -\frac{g}{\Omega_c}\hat{y}$$

which is just the guiding center drift of ions acted on by the gravitational force. Here $\Omega_c = qB/M$ is the ion Larmor frequncy. We can obtain a similar equation for the electrons that drift in the opposite direction. However, in the limit $m/M \rightarrow 0$, the electron contribution can be ignored.



Introduce now a small distrubance so that the boundary becames rippled (lower diagram of figure). Because the **g** × **B** drift is massdependent, the ions will drift faster than the elctrons. hence it can be easily deduced theat the drift υ_0 of the ions over the rippled surface will casue the charges to build up as shown. This charge separation produces an electric field E_1 and sicne the charges change sign between the minimum and the maximum of the ripples, $\mathbf{E}_1 \times \mathbf{B}$ is in the x-direction at the minimum and in the -x-direction at the peaks. The amplitude of the ripple will thus grow larger and the boundary becomes unstable.