Waves in Plasma

The notes given here, closely follow pgs. 79-154 of Chen, "Introduction to Plasma Physics and Controlled Fusion", (Plenum, NY, 1985).

1 Introduction

Plasma is a complex fluid that supports many plasma wave modes. Restoring forces include kinetic pressure and electric and magnetic forces. Wave phenomena are important for heating plasmas, instabilities and diagnostics etc.

In vacuum, there is only one wave mode - the electromagnetic wave with ω/k = c and having oscillating **E** and **B** components perpendicular to k. In air, both sound waves and electromagnetic waves propagate. In plasma, both electrostatic waves and electromagnetic waves will propagate. In the former case, the electric field perturbation associated with the wave is parallel to the wave propoagation direction $\mathbf{E}//\mathbf{k}$ so that there are no magnetic perturbations associated with the wave:

$$\nabla \times \mathbf{E} = ik \times \mathbf{E} = i\omega\mathbf{B} = 0$$

While in air, sound waves propagate through collisions, in a highy ionized plasma, these collisions occur through the wave electric fields.

There are a great variety of possible plasma waves modes, since the wave phase velocity depends on both the wave frequency and its angle of propagation with respect to the background magnetic field. Important characteristic frequencies are ω_{pe}, ω_{ce} and ω_{ci}.

2 Waves

Any sinusoidally oscillating quantity – say, the density n – can be represented as follows:

$$n = n_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$$

where, in Cartesian coordinates,

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$$

Here n_0 is a constant defining the amplitude of the wave, **k** is called the propagation constant. If the wave propages in the x direction, **k** has only an x component:

$$n = \bar{n} e^{i(kx - \omega t})$$

By convention, the exponential notation means that the real part of the expression is to be taken as the measurable quantity. Let us choose $\bar{n}$ to be real; we shall see that this corresponds to a choice of the origins of x and t. The real part of n is then

$$\mathrm{Re}(n) = \bar{n}\cos(kx - \omega t)$$

The exponential notation is useful for analysis of linear systems where Fourier synthesis and superposition are valid.

2.1 Phase velocity

The phase velocity is the velocity on the wave of a point of constant phase. Thus $(d/dt)(kx - \omega t) = 0$ or

$$\frac{dx}{dt} = \frac{\omega}{t} = v_\phi$$

is the phase velocity.

Consider now another oscillating quantity in the wave, say the electric field **E.** Since we have already chosen the phase of n to be zero, we msut alow E to have a different phase δ :

$$\mathbf{E} = \bar{\mathbf{E}}\cos(kx - \omega t + \delta) \text{ or } \mathbf{E} = \bar{\mathbf{E}}e^{i(kx-\omega t+\delta)}$$

where $\bar{\mathbf{E}}$ is a real, constant vector.

It is customary to incorporate the phase information into $\bar{\mathbf{E}}$ by allowing $\bar{\mathbf{E}}$ to be complex. We can write

$$\mathbf{E} = \bar{\mathbf{E}}e^{i\delta}e^{i(kx-\omega t)} \equiv \bar{\mathbf{E}}_c e^{i(kx-\omega t)}$$

Where $\bar{\mathbf{E}}_c$ is a complex amplitude.

$$\tan\delta = \frac{\mathrm{Im}(\bar{\mathbf{E}}_c)}{\mathrm{Re}(\bar{\mathbf{E}}_c)}$$

From now on, we shall assume that all amplitudes are complex and drop the subscript c. Any oscillating quantity g will be wriiten

$$\mathbf{g} = \mathbf{g}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

so that **g** can stand for either the complex amplitude or the entire expression. There can be no confusion, because in linear wave theory the same exponential factor will occur on both sides of any equation and can be cancelled out.

2.2 Group velocity

Information is usually encoded on a carrier wave as either a modulation of its phase or amplitude (or polarization). A simple amplitude modulated wave can be constructed by combining two carriers of slightly different frequency ω-dω and ω+dω. The resulting beat pattern (the information) travels at the *group velocity:*

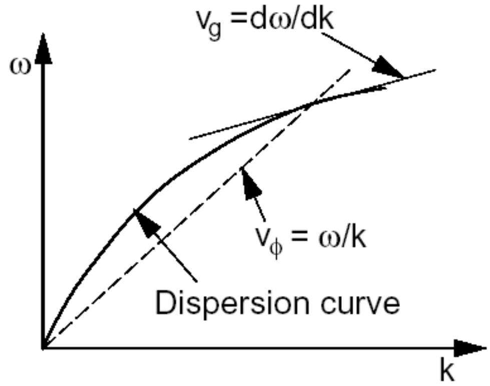

Figure 1:

$$v_g = \frac{d\omega}{dk} < c$$

Let these waves be

$$\begin{aligned} E_1 &= E_0 \cos[(k + \Delta k)x - (\omega + \Delta\omega)t] \\ E_2 &= E_0 \cos[(k - \Delta k)x - (\omega - \Delta\omega)t] \end{aligned}$$

$$E_1 + E_2 = 2E_0 \cos[(\Delta k)x - (\Delta\omega)t] \cos(kx - \omega t)$$

This is a sinusoidally modulated wave. The envelope of the wave, given by $(\Delta k)x-(\Delta\omega)t$, is what carries information; it travels at velocity $\Delta\omega/\Delta k$. Taken the limit $\Delta\omega \to 0$, we thus define the group velocity.

The group velocity is closely related to the concept of Poynting flux. The distinction between phase and group velocities is shown

schematically as follows:

3 Plasma Oscillations

If the electrons in plasma are displaced froma uniform background of ions, electric fields will be built in such a direction as to restore the neutrality of the plasma by pulling the electrons back to their original positions. Becasue of their inertia, the electrons will overshoot and oscillate around their equilibrium positions with a characteristic frequency known as the *plasma frequency.*

Assumptions:

1. There is no magnetic field
2. There are no thermal motions ($kT = 0$)
3. the ions are fixed in space in a uniform distribution;
4. the plasma is infinite in extent
5. the electron motions occur only in the x direction. ($\nabla = \hat{x}\frac{\partial}{\partial x}; \mathbf{E} = E\hat{x}$)

There is, therefore, no fluctating magnetic field; this is an electrostatic oscillation.

The electron equations of motion and continuity are

$$\begin{aligned} mn_e[\frac{\partial \boldsymbol{v}_e}{\partial t} + (\boldsymbol{v}_e \cdot \nabla)\boldsymbol{v}_e] &= -en_e\mathbf{E} \\ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \boldsymbol{v}_e) &= 0 \end{aligned} \tag{1}$$

We use Poisson's equation to find $\mathbf{E}$:

$$\epsilon_0 \nabla \cdot \mathbf{E} = e(n_i - n_e)$$

Linearization: We separate the dependent variables into two parts: an "equilibrium" part indicated by a subscript 0, and a *small* "perturbation" part indicated by a subscript 1:

$$n_e = n_0 + n_1 \quad \boldsymbol{v}_e = \boldsymbol{v}_0 + \boldsymbol{v}_1 \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

$$\begin{aligned} m[\frac{\partial \boldsymbol{v}_1}{\partial t}] &= -e\mathbf{E}_1 \\ \frac{\partial n_1}{\partial t} + n_0 \nabla \cdot \boldsymbol{v}_1 &= 0 \\ \epsilon_0 \nabla \cdot \mathbf{E}_1 &= en_i \end{aligned}$$

where we have already dropped high order perturbation terms and assumed a uniform neutral plasma at equilibrium.

The oscillating quantities are assumed to behave sinusoidally:

$$\begin{aligned} \boldsymbol{v}_1 &= v_1 e^{i(kx-\omega t)}\hat{x} \\ n_1 &= n_1 e^{i(kx-\omega t)} \\ \mathbf{E} &= E e^{i(kx-\omega t)}\hat{x} \end{aligned}$$

Use

$$\frac{\partial}{\partial t} \longleftrightarrow -i\omega \quad \nabla \longleftrightarrow i\mathbf{k}$$

$\Rightarrow$

$$\begin{aligned} -im\omega v_1 &= -eE_1 \\ -i\omega n_1 &= -n_0 ikv_1 \\ ik\epsilon_0 E_1 &= -en_1 \end{aligned}$$

Eliminating n_1 and E_1 :

$$-im\omega v_1 = -i\frac{n_0 e^2}{\epsilon_0 \omega} v_1$$

If v_1 does not vanish, we must have

$$\omega_p^2 = (\frac{n_0 e^2}{\epsilon_0 m})^{1/2}$$

– *Plasma frequency*

This frequency, denpending only on the plasma density, is one of the fundamental parameters of a plasma. Because of the smallness of m, the plasma frequency is usually very high. For instance, in plasma of density $n = 10^{18}$ m^{-3}, we have $f_p = \omega_p/2\pi \simeq 9$ GHz.

In particlular, ω does not depend on k,

$$v_g = \frac{d\omega}{dk} = 0$$

The disturbance does not propagate! They can be pictured as independent oscillators.

4 Electron Plasma Waves

There is another effect that can cause plasma oscillations to propagate, and that is thermal motion. Electrons streaming into adjacent layers of plasma with their thermal velocities will carry information about what is happening in the oscillating region. The plasma oscillation can the properly be called a *plasma wave.*

We can easily treat this effect by adding a term -∇p_e to the equation of motion (equation 1).

For one-dimension, $N = 1$ (freedom), $\gamma = (N+2)/N = 3$

$$\nabla p_e = 3kT_e \nabla n_e = 3kT_e \nabla (n_0 + n_1) = 3kT_e \frac{\partial n_1}{\partial x}\hat{x}$$

and the linearized equation of motion is

$$\begin{aligned} mn_0 \frac{\partial v_1}{\partial t} &= -en_0 E_1 - 3kT_e \frac{\partial n_1}{\partial x} \\ &\Rightarrow \; im\omega n_0 v_1 = -en_0 E_1 - 3kT_e ikn_1 \end{aligned}$$

Equations about E_1 and n_1 are the same as above, and we have

$$im\omega n_0 \upsilon_1 = [en_0(\frac{-e}{ik\epsilon_0}) + 3kT_e ik]\frac{n_0 ik}{i\omega}\upsilon_1$$

$$\omega^2 \upsilon_1 = (\frac{n_0 e^2}{\epsilon_0 m} + \frac{3kT_e}{m}k^2)\upsilon_1$$

$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 \upsilon_{th}^2$$

where $\upsilon_{th}^2 = 2kT_e/m$. The frequency now depend on k, and the group velocity is finite:

$$\upsilon_g = \frac{d\omega}{dk} = \frac{3}{2}\frac{k}{\omega}\upsilon_{th}^2 = \frac{3}{2}\frac{\upsilon_{th}^2}{\upsilon_\phi}$$

Note:

1. $\upsilon_g < c$ (why?)
2. At large k (small λ), information travels essentially at the thermal velocity. At small k, infomation traves more slowly than υ_{th} even though υ_ϕ is greather than υ_{th}. This is because the density gradient is small at large λ, and thermal motions carry very little net momentum into adjacent layers.

Experiments to test the theory:

1. Looney and Brown, 1954
2. A more recent experiment: Barrett, Jones, and Frankling,

5 Sound Waves

As an introduction ion waves, let us briefly review the theory of sound waves in ordinary air.

$$\begin{aligned}
\rho[\frac{\partial \boldsymbol{\upsilon}}{\partial t} + (\boldsymbol{\upsilon}\cdot\nabla)\boldsymbol{\upsilon}] &= -\nabla p = -\frac{\gamma p}{\rho}\nabla\rho \\
\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{\upsilon}) &= 0
\end{aligned}$$

Linearizing about a stationary equilibrium with uniform p_0 and ρ_0, we have

$$\begin{aligned}
-i\omega\rho_0\boldsymbol{\upsilon}_1 &= -\frac{\gamma p_0}{\rho_0} i\mathbf{k}\rho_1 \\
-i\omega\rho_1 + \rho_0 i\mathbf{k}\cdot\boldsymbol{\upsilon}_1 &= 0
\end{aligned}$$

For a plane wave with $\mathbf{k} = k\hat{x}, \boldsymbol{v} = v\hat{x}$, we find, upon elimianting ρ_1

$$-i\omega\rho_0 v_1 = -\frac{\gamma p_0}{\rho_0} ik \frac{\rho_0 ikv_1}{i\omega}$$

$$\omega^2 v_1 = k^2 \frac{\gamma p_0}{\rho_0} v_1$$

or

$$\frac{\omega}{k} = (\frac{\gamma p_0}{\rho_0})^{1/2} = (\frac{\gamma kT}{M})^{1/2} = c_s$$

This is the expression for the velocity c_s of sound waves in a neutral gas. The waves are pressure waves propagationg from one layer to the next by collisions among the air molecules. In a plasma with no neutrals and few collisions, an analogous phenomenon occurs. This is called an ion acoustic wave, or, simply an ion wave.

6 Ion Waves

Ions can still transmit vibrations to each other in the absence of collisions because of their charge. Acoustic waves can occur through the intermediary of electric field. Since the motion of massive ions will be involved, these will be low-frequency oscillations, and we can use the plasma approximation, i.e. $n_i = n_e = n$ insteady of Possion's equation. The ion fluid equation in the absence of a magnetic field is

$$Mn[\frac{\partial \boldsymbol{v}_i}{\partial t} + (\boldsymbol{v}_i \cdot \nabla)\boldsymbol{v}_i] = en\mathbf{E} - \nabla p = -en\nabla\phi - \gamma_i kT_i \nabla n \tag{2}$$

Linearizing and assuming plane waves, we have

$$-i\omega M n n_0 v_{i1} = -en_0 ik\phi_1 - \gamma_i kT_i ikn_1$$

As for the electrons, we may assume m = 0. The balance of forces on electrons, therefore, requires

$$n_e = n = n_0 \exp(e\phi_1/kT_e) = n_0(1 + e\phi_1/kT_e + ...)$$

The perturbation in density of electrons, and therefore, of ions, is then

$$n_1 = n_0 \frac{e\phi_1}{kT_e}$$

The linearized ion equation of continuity

$$i\omega n_1 = n_0 ikv_{i1}$$

Substituting for ϕ_1 and n_1 in terms of v_{i1}

$$i\omega M n_0 v_{i1} = (en_0 ik\frac{kT_e}{en_0} + \gamma_i kT_i ik)\frac{n_0 ikv_{i1}}{i\omega}$$

$$\omega^2 = k^2(\frac{kT_e}{M} + \frac{\gamma_i kT_i}{M})$$

$$\frac{\omega}{k} = (\frac{kT_e + \gamma_i kT_i}{M})^{1/2} \equiv v_s$$

This is the dispersion relation for ion acoustiv waves; v_s is the sound speed in a plasma.

Note:

1. Since the ions suffer one-dimensional compressons in the plane waves, $\gamma_i = 3$.The elctrons move so fast relative to these waves that they have time to equalize their temperature everywhere; therefore, the electrons are isothermal, and $\gamma_e = 1$.
2. $v_g = v_\phi$
3. When $kT_i \to 0$, ion waves still exist.

7 Validity of the Plasma Approximation

In deriving the velocity of ion waves, we used the neutrality conditions $n_i = n_e$ while allowing $\mathbf{E}$ to be finite. To see what error was engendered in the process, we now allow n_i to differ from n_e and use the linearized Possion equation:

$$\epsilon_0 \nabla \cdot \mathbf{E}_1 = \epsilon_0 k^2 \phi_1 = e(n_{i1} - n_{e1})$$

The electron density is given by the linearized Boltzmann relation:

$$n_1 = n_0 \frac{e\phi_1}{kT_e}$$

Thus

$$\begin{aligned} \epsilon_0 \phi_1 (k^2 + \frac{n_0 e^2}{\epsilon_0 kT_e}) &= en_{i1} \\ \epsilon_0 \phi_1 (k^2 \lambda_D^2 + 1) &= en_{i1} \lambda_D^2 \end{aligned}$$

The ion density is given by the linearized ion continuity equation:

$$n_{i1} = \frac{k}{\omega} n_0 v_{i1}$$

Inserting into the ion equation of motion (Equation 2)

$$i\omega M n_{n0} v_{i1} = (\frac{en_0 ik}{\epsilon_0} \frac{e\lambda_D^2}{1 + k^2 \lambda_D^2} + \gamma_i kT_i ik) \frac{k}{\omega} n_0 v_{i1}$$

$$\omega^2 = \frac{k^2}{M}(\frac{n_0 e^2 \epsilon_0^{-1} \lambda_D^2}{1+k^2\lambda_D^2} + \gamma_i k T_i) \tag{3}$$

$$\frac{\omega}{k} = (\frac{kT_e}{M}\frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i k T_i}{M})^{1/2}$$

This is the same as we obtained previously except for the factor $1 + k^2\lambda_D^2$. Our assumption $n_i = n_e$ has given rise to an error of order $k^2\lambda_D^2 = (2\pi\lambda_D/\lambda)^2$. Since λ_D is very samll in most experiments, the plasma approximation is valid for all except the shortest wavelength waves.

8 Comparision of Ion and Electron Waves

If we consider these short-wavelength waves by taking $k^2\lambda_D^2 >> 1$, Eq 3 becomes

$$\omega^2 = k^2 \frac{n_0 e^2}{\epsilon_0 M k^2} = \frac{n_0 e^2}{\epsilon_0 M} \equiv \Omega_p^2$$

We have, for simplicity, also taken the limit $T_i \to 0$. Here Ω_p is the ion plasma frequency. For high frequencies the ion acoustic wave turns into a constant-frequency wave. There is thus a complementary behavior between electron plasma waves and ion acoustic waves:

electron plasma waves: basically constant frequency, but become constant velocity at large k

ion acoustic waves: basically constant velocity, but become constant frequency at large k.

9 Electrostatic Electron Oscillations Perpendicular to B

Up to now, we have assumed B = 0,. When a magnetic field exists, may more types of waves are possible. We shall examine only the simplest cases, starting with high-frequency, electrostatic, electron oscillations propagating at right angles to the magnetic field.

Some definitions:

Parallel / perpendicular: denote the direction of $\mathbf{k}$ relative to the undisturbed magnetic field $\mathbf{B}_0$.

Longitudianl / transverse: refer to the direction of of $\mathbf{k}$ relative to the oscillating electric field $\mathbf{E}_1$.

electrostatic / electromagnetic: the oscillating magnetic field $\mathbf{B}_1 = 0$ / $\mathbf{B}_1 \neq 0$

The last two sets of terms are related by Maxwell's equation

$$\nabla \times \mathbf{E}_1 = -\dot{\mathbf{B}}_1$$

or

$$\mathbf{k} \times \mathbf{E}_1 = \omega \mathbf{B}_1$$

If a wave is longitudianl, $\mathbf{k} \times \mathbf{E}_1$ vanishes, and the wave is also electrostatic. If the wave is transevese, $\mathbf{B}_1$ is finite, and the wave is electromagnetic. It is of course possibel for $\mathbf{k}$ to be at an arbitrary angle to $\mathbf{B}_0$ and $\mathbf{E}_1$, then one would have a mixture of the principal modes presented here.

Electron oscillations perpendicular to $\mathbf{B}_0$.

Assumptions:

1. Ions fixed
2. $kT_e = 0$

The motion of electrons in then governed by the following linearized equations:

$$\begin{aligned} m\frac{\partial \boldsymbol{v}_{e1}}{\partial t} &= -e(\mathbf{E}_1 + \boldsymbol{v}_{e1} \times \mathbf{B}_0) \\ \frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \boldsymbol{v}_{e1} &= 0 \\ \epsilon_0 \nabla \cdot \mathbf{E}_1 &= -en_{e1} \end{aligned}$$

We shall consider only longitudianl waves with $\mathbf{k}//\mathbf{E}_1$. Without loss of generality, we can choose the x axis to lie along $\mathbf{k}$ and $\mathbf{E}_1$, and the z axis lie along $\mathbf{B}_0$.

Dropping the subscripts 1 and e and separating the momentum equations into components

$$\begin{aligned} -i\omega m v_x &= -eE - ev_y B_0 \\ -i\omega v_y &= ev_x B_0 \\ -i\omega m v_z &= 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} i\omega m v_x &= eE + eB_0 \frac{ieB_0}{m\omega} v_x \\ v_x &= \frac{eE/im\omega}{1 - \omega_c^2/\omega^2} \end{aligned}$$

The linearized continuity equations:

$$n_1 = \frac{k}{\omega} n_0 v_x$$

Linearizing the Possion's equation and using the last two results

$$\begin{aligned} ik\epsilon_0 E &= -e\frac{k}{\omega}n_0\frac{eE}{im\omega}(1-\frac{\omega_c^2}{\omega^2})^{-1} \\ (1-\frac{\omega_c^2}{\omega^2})E &= \frac{\omega_p^2}{\omega^2}E \end{aligned}$$

The dispersion relation is therefore

$$\omega^2 = \omega_p^2 + \omega_c^2 \equiv \omega_h^2$$

The frequency ω_h is called the upper hybrid frequency. Electrostatic electron waves across **B** have this frequency, while whose along **B** are the usual plasma oscillation with $\omega = \omega_p$. The group velocity is again zero as long as thermal motions are neglected.

Physical picture: There are two restoring forces acting on the electrons: the electrostatic field and the Lorentz force. The increased restoring force makes the frequency larger than that of a plasma oscillation.

10 Electrostatic Ion Waves Perpendicular to B

We next consider what happens to the ion acoustic wave when **k** is perpendicular to $\mathbf{B}_0$.

Assumptions:

1. n_0 and B_0 constant and uniform and $v_0 = E_0 = 0$.
2. $T_i = 0$
3. Electrostaic waves with $\mathbf{k} \times \mathbf{E} = 0$, so that $\mathbf{E} = -\nabla\phi$
4. $\mathbf{E} = E_1\hat{x}$ and $\nabla = ik\hat{x}$
5. let **k** be almost perpendicualr to $\mathbf{B}_0$, allowing the eclectrons to preserve charge neutrality by flowing along the B lines.

For the ion equation of motion, we have

$$M\frac{\partial \boldsymbol{v}_{i1}}{\partial t} = -e\nabla\phi_1 + e\boldsymbol{v}_{i1} \times \mathbf{B}_0$$

Assuming plane waves propagating in the x direction and separating into components, we have

$$\begin{aligned} -i\omega M v_{ix} &= -eik\phi_1 + ev_{iy}B_0 \\ -i\omega M v_{iy} &= -ev_{ix}B_0 \end{aligned}$$

Solving as before, we find

$$\upsilon_{ix} = \frac{ek}{M\omega}\phi_1(1 - \frac{\Omega_c^2}{\omega^2})^{-1}$$

where $\Omega_c = eB_0/M$ is the ion cyclotron frequency. The ion equation of continuity yields, as usual.

$$n_1 = \frac{k}{\omega}n_0\upsilon_{ix}$$

Assuming the electrons can move along $\mathbf{B}_0$,we can use the Boltzmann relation for electrons.

$$\frac{n_{e1}}{n_0} = \frac{e\phi_1}{kT_e}$$

The plasma approximation $n_i = n_e$ now closes the system of equations.

$$(1 - \frac{\Omega_c^2}{\omega^2})\upsilon_{ix} = \frac{ek}{M\omega}\frac{k_BT_e}{en_0}\frac{n_0k}{\omega}\upsilon_{ix}$$

$$\omega^2 = \Omega_c^2 + k^2\frac{k_BT_e}{M}$$

$$\omega^2 = \Omega_c^2 + k^2\upsilon_s^2$$

This is the dispersion relation for electrostatic ion cyclotron waves.

Physical picture: the ions undergo an acoustic-type oscillation, but the Lorentz force constitutes a new restoring force giving rise to the Ω_c^2.

11 The Lower Hybrid Frequency

We now consider what happens when θ is exactly $\pi/2$, and the electrons are not allowed to preserve charge neutrality by flowing along the lines of force. Insteady of obeying Boltzmann's relation, they will obey the full equation of motion.

The ion equation of motion us unchanged:

$$\upsilon_{ix} = \frac{ek}{M\omega}\phi_1(1 - \frac{\Omega_c^2}{\omega^2})^{-1}$$

By changing e to -e, M to m, and Ω_c to -ω_c, we can write down the result for electrons, with $\mathrm{T}_e = 0$

$$\upsilon_{ex} = \frac{-ek}{m\omega}\phi_1(1 - \frac{\omega_c^2}{\omega^2})^{-1}$$

The equations of continuity give

$$n_{i1} = n_0\frac{k}{\omega}\upsilon_{i1} \quad n_{e1} = n_0\frac{k}{\omega}\upsilon_{e1}$$

The plasma approximation $n_i = n_e$, then requires $v_{i1} = v_{e1}$

$$M(1 - \frac{\Omega_c^2}{\omega^2}) = -m(1 - \frac{\omega_c^2}{\omega^2})$$

$$\omega^2(M + m) = m\omega_c^2 + M\Omega_c^2 = e^2B^2(\frac{1}{m} + \frac{1}{M})$$

$$\omega^2 = \frac{e^2B^2}{Mm} = \omega_c\Omega_c$$

$$\omega = (\omega_c\Omega_c)^{1/2} \equiv \omega_l$$

This is called the lower hybrid frequency. If we had used Poisson's equation instead of the plasma approximation, we would have obtained:

$$\frac{1}{\omega_l^2} = \frac{1}{\omega_c\Omega_c} + \frac{1}{\Omega_p^2}$$

In low-density plasmas the latter term actually dominates. The plasma approximation is not valid at such frequencies. Lower hybrid oscillations can be observed only is θ is very close to $\pi/2$.