1 Plasma Kinetic Theory

1.1 Theoretical Hierarchy of Plasma Physics

In any macroscopic physical system containing many individual particles, there are basically three levels of description: the *exact microscopic* description, *kinetic theory*, and the *macroscopic or fluid* description. In a microscopic description, one imagines writing down Newton's law, $\mathbf{F} = m\mathbf{a}$, for something like 10^{20} particles and solving for all 10^{20} interacting trajectories. Such a description is in principle exact, classically. Laplace was able to boast, "Give me the initial data on the particles and I'll predict the future of the universe", even though he knew the system was insoluble in practice. It is still unimaginable today, even by the most advanced computers. Even the initial data itself is beyond the magnitude of imaginable storage deveices. Moreover, if solutions were known, they would be mostly irrelevant information requiring another unimaginably advanced computers to distill into useful form. When the sensitivity of the exact solution to miniscule initial condition erros is considered – the modern study of *chaos* – the situation becomes even more absurd. Nonetheless, the microscopic description is useful as a formal starting point from which to derive soluble, practical descriptions.

The microscopic theory passed to kinetic theory by the application of statistical, probability concepts. SInce one is not interested in all the microscopic particle data, one considers statistical ensembles of systems. By averaging out the microscopic information in the exact theory, one obtains statistical, kinetic equations. When possible, these are reduced further to give equations for the one-paricle (i.e., electron or ion) distribuition fucntions. Examples of kinetic equations are the Vlasov equation and the Boltzmann equation. Althouth the precise locations of individual particles are lost in kinetic theory, detailed knowledge of particle motion is requred. In this sense kinetic theory is still microscopic, even though statistical averages have been employed. Finally, in some cases, it is possible to reduce kinetic theory even further. Here, one has only macroscopic quantities such as density, temperature, and fluid velocity, and closed equations giveing their evolution in space and time. No knowlege of individual paricle motion is required to describe observable phenomena.

As an introductory course, in this chapter will not be based on a formulation of the microscopic world originated by Klimontovich. Instead, we follow Chen's approach (1984) to "derive" kinetic theory equaitons.

1.2 Kinetic Theory

A fluid model description of plasma waves and oscillations is frequently inadequate. For these, we need to consider the velocity distribution fuction $f(\boldsymbol{v})$; this treatment is called kinetic theory. In fluid theory, the dependent variables are functions of only four independent variables: x, y, z and t. This is possible because the velocity distribution of each epecies is assumed to be Maxwellian everywhere and can therefore be uniquely specified by only one number, the temperature T. However, if the velocity of a significant number of charged particles (typically the thermal velocity) is near to the phase velocity of waves, the wave-particle interaction is significantly different from that described by the fluid equations. A proper description of this interaction must be based upon the dynamics of the particles' phase space distribution function in which the particles' velocity and position are independent variables. Such a description is provided by the kinetic theory equations for a plasma with self-consistent fields.

1.3 Distribution Functions

A plasma is an ensemble of particles electrons e, ions i and neutrals n with different positions $\mathbf{r}$ and velocities $\mathbf{v}$ which move under the influence of external forces (electromagnetic fields, gravity) and internal collision processes (ionization, Coulomb, charge exchange etc.)

However, what we observe is some "average" macroscopic plasma parameters such as j - current density, n_e - electron density, P - pressure, T_i - ion temperature etc. These parameters are macrosocopic averages over the distribution of particle velocities and/or positions.

1.4 Phase Space

Consider a single particle of species α. It can be described by a position vector

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

in configuration space and a velocity vector

$$\boldsymbol{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

in velocity space. The coordinates $(\mathbf{r}, \boldsymbol{v})$ define the particle position in phase space.

For multi-particle systems, we introduce the distribution fuction $f_\alpha(\mathbf{r}, \boldsymbol{v}, t)$ for species α defined such that

$$f_\alpha(\mathbf{r}, \boldsymbol{v}, t) d\mathbf{r} d\boldsymbol{v} = dN(\mathbf{r}, \boldsymbol{v}, t)$$

is the number of particles in the element of volume $dV = d\mathbf{r}d\boldsymbol{v}$ in phase space. Here $d\mathbf{r} \equiv d^3r \equiv dxdydz$ and $d\boldsymbol{v} \equiv d^3v = dv_x dv_y dv_z$. $f_\alpha(\mathbf{r}, \boldsymbol{v}, t)$ is a positive finite function that decreases to zero as $|v|$ becomes large.

The element $d\mathbf{r}$ must not be so small that it doesn't contain a statistically significant number of particles. This allows $f_\alpha(\mathbf{r}, \boldsymbol{v}, t)$ to be approximated by a continuous function.

Some definitions:

- If f_α depends on $\mathbf{r}$, the distribution is *inhomogneous*
- If f_α is independent of $\mathbf{r}$, the distribution is *homogneous*

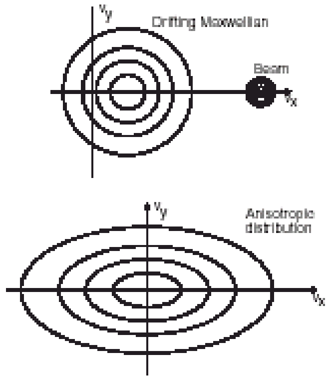

Figure 1:

- If f_α depends on the direction of $\boldsymbol{v}$, the distribution is *anisotropic*
- If f_α is independent of the direction of $\boldsymbol{v}$, the distribution is *isotropic*
- A plasma in thermal equilbrium is characterized by a homogeneous, isotropic and time-independent distribution funciton.

One type of contour map can be made for f if we consider $f(\boldsymbol{v})$ at a given point in space, which are very useful for getting a preliminary idea of how the plasma behaves. For instance, if the motion is two dimensional, the contours $f(v_x, v_y)$ will be circles if f is isotropic in v_x, v_y. An anisotropic distribution would have elliptical contours. A drifting Maxwellian would have cicular contours dispaced from the origin, and a beam of particles traveling in the x direction would show up as a separate spike.

1.5 Equation of Kinetic Theory

Vlasov Equation

Treat particles as moving in 6-D phase $\mathbf{r}$ position, $\boldsymbol{v}$ velocity. At any instant a particle occupies a unique position in phase space $(\mathbf{r}, \boldsymbol{v})$.

Consider an elemental volume $d\mathbf{r}d\boldsymbol{v}$ of phase space $[dxdydzdv_xdv_ydv_z]$, at $(\mathbf{r}, \boldsymbol{v})$. Write down an equation that is conservation of particles for this volume

$$
\begin{aligned}
-\frac{\partial}{\partial t}(fd^3rd^3\boldsymbol{v}) = [\quad & v_x f(x + dx\hat{x}, \boldsymbol{v}) - v_x f(x, \boldsymbol{v})]dydzd^3\boldsymbol{v} \\
& + \text{same for dy,dz} \\
& +[a_x f(\mathbf{r}, \boldsymbol{v} + dv_x\hat{v}_x) - a_x f(\mathbf{r}, \boldsymbol{v})]d^3rdv_ydv_z \\
& +\text{same for d}v_y\text{,d}v_z
\end{aligned}
$$

a is 'velocity' in velocity i.e. acceleration.
Divide through by $d^3rd^3\boldsymbol{v}$ and take limit

$$
\begin{aligned}
-\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}(v_x f) + \frac{\partial}{\partial y}(v_y f) + \frac{\partial}{\partial z}(v_z f) \\
&\quad + \frac{\partial}{\partial v_x}(a_x f) + \frac{\partial}{\partial v_y}(a_y f) + \frac{\partial}{\partial v_z}(a_z f) \\
&= \nabla \cdot (\boldsymbol{v} f) + \nabla_v \cdot (\mathbf{a} f)
\end{aligned}
$$

[Notation: use $\partial/\partial\mathbf{r} \leftrightarrow \nabla$ as usual in (x,y,z) space; $\partial/\partial\boldsymbol{v} \leftrightarrow \nabla_v$

$$
\frac{\partial}{\partial \boldsymbol{v}} = \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z}
$$

]

Take this simple continuity equation in phase space and expand:

$$
\frac{\partial f}{\partial t} + (\nabla \cdot \boldsymbol{v})f + (\boldsymbol{v} \cdot \nabla)f + (\nabla_v \cdot \mathbf{a})f + (\mathbf{a} \cdot \nabla_v)f = 0
$$

Recognize that ∇ means here $\partial/\partial x$ etc keeping v constant. so that $\nabla \cdot \boldsymbol{v} = 0$ by definition. So

$$
\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -f \ \ (\nabla_v \cdot \mathbf{a})
$$

The Vlasov-Maxwell Plasma Model

Now we want to couple this equation with Maxwell's equations for the fields self-consistently to the plasma charges and currents as determined from the appropriate moments of the distribution funciton, f,

$$
\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}; \ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}; \ \nabla \cdot \mathbf{B} = 0
\end{aligned}
$$

Coupling is completed via charge & current densities

$$
\rho = \sum_j q_j n_j = \sum_j q_j \int f_j d^3\boldsymbol{v}
$$

$$
\mathbf{j} = \sum_j q_j n_j \mathbf{V}_j = \sum_j q_j \int f_j \boldsymbol{v} d^3\boldsymbol{v}
$$

and the Lorentz force

$$
\mathbf{a} = \frac{q}{m}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B})
$$

Actually we don't want to use the **E** retaining all the local effects of individual particles. We want a smoothed out field. Ensemble averaged **E**.

Evaluate

$$\begin{aligned}\nabla_v \cdot \mathbf{a} &= \nabla_v \cdot \frac{q}{m}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) = \frac{q}{m}\nabla_v \cdot (\boldsymbol{v} \times \mathbf{B}) \\ &= \frac{q}{m}\mathbf{B}\cdot(\nabla_v \times \boldsymbol{v}) = 0\end{aligned}$$

So RHS is zero. However, in the use of smoothed out $\mathbf{E}$ we have ignored local effect of one particle on another due to the graininess. This is collisions.

Summary:
Coupled to Vlasov equation for each particle species we have Maxwell's equations.

Vlasov-Maxwell Equations

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_j}{\partial \mathbf{r}} + \frac{q_j}{m_j}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f_j}{\partial \boldsymbol{v}} = 0 \tag{1}$$

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t};\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0};\ \nabla \cdot \mathbf{B} = 0\end{aligned}$$

Coupling is completed via charge & current densities

$$\rho = \sum_j q_j n_j = \sum_j q_j \int f_j d^3\boldsymbol{v}$$

$$\mathbf{j} = \sum_j q_j n_j \mathbf{V}_j = \sum_j q_j \int f_j \boldsymbol{v} d^3\boldsymbol{v}$$

Describe phenomena in which collisions are not important, keeping track of the (statistically averaged) particle distribution fucntion.

Plasma waves are the most important phenomena covered by the Vlasov-Maxwell equations.

6-dimensional, nonlinear, time-dependent, integro-differential equations!

Boltzmann Equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = (\frac{\partial f}{\partial t})_{collisions}$$

When there collisions with neutral atoms, the collision term can be approximated by

$$(\frac{\partial f}{\partial t})_c = \frac{f_n - f}{\tau}$$

where f_n is the distribution function of the neutral atoms, and τ is a constant collision time. This is called a *Krook collision term.* When there are Coulomb collisions,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = -\frac{\partial}{\partial \boldsymbol{v}} \cdot (f < \Delta \boldsymbol{v} >) + \frac{1}{2} \frac{\partial^2}{\partial \boldsymbol{v} \partial \boldsymbol{v}} : (f < \Delta \boldsymbol{v} \Delta \boldsymbol{v} >)$$

This is called the *Fokker-Planck equation*, it takes into account binary Coulomb collisions only. Here $\Delta \boldsymbol{v}$ is the change of velocity in a collision. The first term describes the frictional force slowing down fast particles and accelerating slow ones. The negative divergence in velocity space describes a narrowing of the distribution. In the second term $< \Delta \boldsymbol{v} \Delta \boldsymbol{v} > / \boldsymbol{\Delta} t$ is a coefficient of diffusion in velocity space. This term then describes the fact that a narrow velocity distribution (e.g. a beam) will broaden as a result collisions. The two terms thus operate in opposite senses, and are in balance for an equilibrium (Maxwellian) distribution.

Vlasov Equation = Boltzmann Eq without collisions.

For electromagnetic forces:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

1.6 General Properties of the Vlasov Model:

1. *Distribution function is constant along particle orbit in phase space:*

$$\frac{d}{dt} f = 0$$

$$\frac{d}{dt} f = \frac{\partial f}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{d\boldsymbol{v}}{dt} \cdot \frac{\partial f}{\partial \boldsymbol{v}}$$

It simply expresses conservation of particles in phase space with mothion due to the smooth self-consistent fields. Note the density in real space, $\int d^3 v f$, can vary.

2. *The Vlasov equation is time reverible:*

Here we simply note that the time reversal transformation,

$$\begin{aligned} t &\rightarrow -t \\ \mathbf{r} &\rightarrow \mathbf{r} \\ \boldsymbol{v} &\rightarrow -\boldsymbol{v} \\ \mathbf{E} &\rightarrow \mathbf{E} \\ \mathbf{B} &\rightarrow -\mathbf{B} \end{aligned}$$

This leaves the equation invariant. This means that starting from some initial state and evolving by the Vlasov model, at any point in time if one reversed the velocity of all the pariticles (which would make the magnetic field, B, reverse) the system would retrace its steps to the initial state. There is no tendency to relax to a thermodynamically favoriable state, or distribution fucntion. The system evolves with reversible, if complicated, dynamics. This point is emphasized further by the following property.

3. *Entropy is Conserved in the Vlasov Model:*
The basic formula for the entropy in non-equlibrium systems is:

$$S = -\int d^3r d^3v f \ln f$$

This a form familiar to students of information theroy, since this is, effectively, the formula, $\sum_i P_i \log_2 P_i$, that Shannon found for the information, in bits, of some signal.

$$\begin{aligned}
\frac{dS}{dt} &= -\int d^3r d^3v \{\frac{\partial f}{\partial t} + \ln f \frac{\partial f}{\partial t}\} \\
&= \int d^3r d^3v \{\boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} + \ln f (\boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}})\} \\
&= \int d^3r d^3v \{\boldsymbol{v} \cdot \frac{\partial}{\partial \mathbf{r}} (f \ln f) + \mathbf{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} (f \ln f)\} \\
&= \int d^3r d^3v \{\frac{\partial}{\partial \mathbf{r}} \cdot (\boldsymbol{v} f \ln f) + \frac{\partial f}{\partial \boldsymbol{v}} \cdot (\mathbf{a} f \ln f)\} \\
&= \int d^3v \int d\mathbf{A}_r \cdot (\boldsymbol{v} f \ln f) + \int d^3r \int d\mathbf{A}_v \cdot (\mathbf{a} f \ln f)\} \qquad (2)
\end{aligned}$$

The last term is zero, assuming that the distribution function, f, vanishes, as $v \to \infty$. The first term is likewise is zero for a bounded system containing a finite number of particles, so that,

$$\frac{dS}{dt} = 0$$

Notes Eq. 2 represent flows of entropy across the boundaries of the system, both in space and velocity. As long as the system boundary conditions prohibit entropy exchange with the outside world, entropy will be conserved.

These properties further emphasize that the strict Vlasov plasma, although capable of undergoing extremely complex behavior, is nonetheless reversible. If the particles were truned around, $\boldsymbol{v} \to -\boldsymbol{v}$, the system would evolve exactly to its initial state.

1.7 The Equivalence of Kinetic Theory and Orbit Theory

The collisionless Boltzman equation (Vlasov) is the root of kinetic theory

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

The base equation of orbit theory is Newton's law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$$

Being a second order differential equation in there dimension, the general solution of Newton's equaiton must contain six constants of intergration, $\alpha_1, ..., \alpha_2$. We write the solution

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(\boldsymbol{\alpha}_1, ..., \alpha_6, t) \\ \boldsymbol{v} &= \boldsymbol{v}(\boldsymbol{\alpha}_1, ..., \alpha_6, t) \end{aligned}$$

In principle, we can formally solve these six scalar equations for the α_i :

$$\alpha_i = \alpha_i(\mathbf{r}, \boldsymbol{v}, t), i = 1 - 6$$

Now any arbitray function of the α_i, $f = f(\boldsymbol{\alpha}_1, ..., \alpha_6)$, is a solution of the Vlasov equation above:

$$\frac{d}{dt} f(\boldsymbol{\alpha}_1, ..., \alpha_6) = \sum_i \frac{\partial f}{\partial \alpha_i}\frac{d\alpha_i}{dt} = 0$$

The result is identically zero because the $\alpha_i's$ are constants.

Thus the general soltuon of the Valsov equation is an arbitray function of the integrals of Newton's law, and the two approaches are equivalent. It is sometimes called Jean's theorem.

1.8 Derivation of the Fluid Equations

The fluid equations we have beening using are simply moments of the Boltzmann (or Vlasov if collisions are ignored) equations. For any $A(\boldsymbol{v})$, the hydrodynamic evolution equation for $< A(\boldsymbol{v}) >$ will be given by

$$\int A(\boldsymbol{v})[\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}}]d^3v = 0$$

Letting $A(\boldsymbol{v}) = v^0 = 1, m\boldsymbol{v}, m\boldsymbol{v}\boldsymbol{v}/2$ we obtain, respectively:

- The equation for the density: equation of continuity
- The equation for the momentum density: equation of motion
- The equation for the kinetic energy density: heat flow equation (energy equation)

1.8.1 Equation of continuity

Let $A(\boldsymbol{v}) = 1$:

$$\int \frac{\partial f}{\partial t} d\boldsymbol{v} + \int \boldsymbol{v} \cdot \nabla f d\boldsymbol{v} + \frac{q}{m} \int (\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = 0$$

The first term gives

$$\int \frac{\partial f}{\partial t} d\boldsymbol{v} = \frac{\partial}{\partial t} \int f d\boldsymbol{v} = \frac{\partial n}{\partial t}$$

Since $\boldsymbol{v}$ is an independent variable

$$\int \boldsymbol{v} \cdot \nabla f d\boldsymbol{v} = \boldsymbol{\nabla} \cdot \int \boldsymbol{v} f d\boldsymbol{v} = \boldsymbol{\nabla} \cdot (n\bar{\boldsymbol{v}}) = \boldsymbol{\nabla} \cdot (n\mathbf{u})$$

where the average $\mathbf{u}$ is the fluid velocity by definition.

$$\int \mathbf{E} \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\mathbf{E}) d\boldsymbol{v} = \int_{S\infty} f\mathbf{E} \cdot d\mathbf{S}$$

The perfect divergence is integrated to give the value of $f\mathbf{E}$ on the surface at $v = \infty$. This vanishes if $f \to 0$ faster than v^{-2} as $v \to \infty$, as is necessary for any distribution with finite energy.

$$\int (\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = \int \frac{\partial}{\partial v} \cdot (f\boldsymbol{v} \times \mathbf{B}) d\boldsymbol{v} - \int f \frac{\partial}{\partial v} \times (\boldsymbol{v} \times \mathbf{B}) d\boldsymbol{v} = 0$$

The first integral can again be converted to a surface integral. For a Maxwellain, f falls faster than any power of v as $v \to \infty$, and the integral therefore vanishes. The second integral vanishes because $\boldsymbol{v} \times \mathbf{B}$ is perpendicular to $\frac{\partial}{\partial v}$. Hence, we get the equation of continuity:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

1.8.2 Fluid equation of motion

Let $A(\boldsymbol{v}) = m\boldsymbol{v}$:

$$m \int \boldsymbol{v} \frac{\partial f}{\partial t} d\boldsymbol{v} + m \int \boldsymbol{v}\boldsymbol{v} \cdot \nabla f d\boldsymbol{v} + q \int \boldsymbol{v}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = 0$$

The first term gives

$$m \int \boldsymbol{v} \frac{\partial f}{\partial t} d\boldsymbol{v} = m \frac{\partial}{\partial t} \int \boldsymbol{v} f d\boldsymbol{v} = m \frac{\partial}{\partial t} (n\mathbf{u})$$

The third integral can be written

$$\begin{aligned} \int \boldsymbol{v}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} \quad = \quad & \int \frac{\partial}{\partial \boldsymbol{v}} \cdot [f\boldsymbol{v}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B})] d\boldsymbol{v} \\ & - \int f\boldsymbol{v} \frac{\partial}{\partial \boldsymbol{v}} \cdot (\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) d\boldsymbol{v} \\ & - \int f(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{v} d\boldsymbol{v} \end{aligned}$$

The first two integrals vanish for the same reasons as before, and $\partial \boldsymbol{v}/\partial \boldsymbol{v}$ is just the identity tensor $\overleftrightarrow{\mathbf{I}}$. We therefore have

$$q \int \boldsymbol{v}(\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} d\boldsymbol{v} = -\, q \int (\mathbf{E} + \boldsymbol{v} \times \mathbf{B}) f d\boldsymbol{v} = -\, qn(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Finally, to evaluate the second integral,

$$\int \boldsymbol{v}\boldsymbol{v} \cdot \nabla f d\boldsymbol{v} = \int \nabla \cdot (f\boldsymbol{v}\boldsymbol{v}) d\boldsymbol{v} = \nabla \cdot \int f\boldsymbol{v}\boldsymbol{v} d\boldsymbol{v} = \boldsymbol{\nabla} \cdot n\overline{\boldsymbol{v}\boldsymbol{v}}$$

Now we may separate $\boldsymbol{v}$ into the average (fluid) velocity $\mathbf{u}$ and a thermal velocity $\mathbf{w}$:

$$\boldsymbol{v} = \mathbf{u} + \mathbf{w}$$

Since $\mathbf{u}$ is already an average, we have

$$\boldsymbol{\nabla} \cdot n\overline{\boldsymbol{v}\boldsymbol{v}} = \nabla \cdot (n\mathbf{u}\mathbf{u}) + \boldsymbol{\nabla} \cdot n\overline{\mathbf{w}\mathbf{w}} + 2\nabla \cdot (n\mathbf{u}\overline{\mathbf{w}})$$

The average $\overline{\mathbf{w}}$ is obviously zero. The quantity $mn\overline{\mathbf{w}\mathbf{w}}$ is precisely what is meant by the stress tensor $\overleftrightarrow{P}$:

$$\overleftrightarrow{P} \equiv mn\overline{\mathbf{w}\mathbf{w}}$$

$$\nabla \cdot (n\mathbf{u}\mathbf{u}) = \mathbf{u}\boldsymbol{\nabla} \cdot (n\mathbf{u}) + n(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$$

Therefore:

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + m\mathbf{u}\boldsymbol{\nabla} \cdot (n\mathbf{u}) + mn(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} \cdot \overleftrightarrow{P} - qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = 0$$

Combining the first two terms with the help of equation of continutiy, we finally obtain teh fluid equation of motion:

$$mn[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \boldsymbol{\nabla} \cdot \overleftrightarrow{P}$$

To treat the flow of energy, we may take $A(\boldsymbol{v}) = m\boldsymbol{v}\boldsymbol{v}/2$. The equation $p\tilde{\ }\rho^{\gamma}$ is a simple form of the heat flow equation for themal conductivity $\kappa = 0$.