1 The Landau Problem and Collisionless Dissipation

The basic effect that plasma waves damp, even in the absence of collisions was first discovered theoretically by Landau and is referred as Landau damping.

1.1 Linearized Electron Plasma Waves: Vlasov's Solution

As an elementary illustration of the use of the Vlasov equation, we shall derive the dispersion relation for electron plasma oscillations, which we treated from the fluid point of view before.

It is evident from the last term in that the Vlasov-Maxwell model equations are nonlinear and hence in general difficult to solve. We focus our attention on the solution of these equations for *small-amplitude perturbations* abourt a given uniform equilibrium so that the equations can be reduced to linear partial differential equations. These form the basis for studying the kinetic theory of linear waves and instabilities in plasmas.

In zero order, we assume a uniform plasma with a distribution f_0,and Let $\mathbf{B}_0 = \mathbf{E}_0 = 0$.

Linearize the Vlasov Eq by supposing

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\boldsymbol{v}) + f_1(\mathbf{r}, \mathbf{v}, t), \quad f_1 \text{ small}$$

The first-order Vlasov equation for electrons is

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} f_1 - \frac{e}{m}\mathbf{E}_1 \cdot \frac{\partial f_0}{\partial t} = 0$$

As before, we assume the ions are massive and fixed and that the waves are plane wave in the x-direction

$$f_1 \to e^{i(kx-\omega t)}$$

Then

$$-i\omega f_1 + ikv_x f_1 - \frac{e}{m}E_x\frac{\partial f_0}{\partial v_x} = 0$$

Solution:

$$f_1 = \frac{ieE_x}{m}\frac{\partial f_0/\partial v_x}{\omega - kv_x}$$

Poisson's equation gives

$$\epsilon_0 \nabla \cdot \mathbf{E}_1 = ik\epsilon_0 E_x = -e\iiint f_1 d^3v$$

Substituting for f_1 :

$$1 = -\frac{e^2}{km\epsilon_0}\iiint \frac{\partial f_0/\partial \upsilon_x}{\omega - k\upsilon_x} d^3\upsilon$$

A normalized fucntion

$$\hat{f}_0 = f_0/n_0$$

If f_0 is a Maxwellian or some other factorable distribution, the integrations over υ_y, υ_z can be carried out easily. What remains is the one dimensional distribution fucntion $\hat{f}_0(\upsilon_x)$.

The dispersion relation is, therefore,

$$1 = \frac{\omega_p^2}{k^2}\int \frac{\partial \hat{f}_0/\partial \upsilon}{\upsilon - (\omega/k)} d\upsilon \tag{1}$$

where we have dropped the subscript x.

Here $\hat{f}_0$ is understood to be a one-dimensional distribution fucntion, the integrations over υ_y, υ_z having been made.

To give an example of this dispersion, cosider the properties of plasma waves in a thermal equilibrium plasma. In this case,

$$\hat{f}_0 = \frac{1}{\sqrt{\pi}\upsilon_e}\exp(-\upsilon^2/\upsilon_e^2)$$

where $\upsilon_e = \sqrt{2KT_e/m_e}$, is the electron thermal velocity. For waves with phase velocitites much larger than the electron thermal speed, $\omega/k >> \upsilon_e$, $k\upsilon_e/\omega$ will be samll for most velocities. [We previously argued that cold-plasma is valid if $\omega/k >> ve$]. The enequality, $k\upsilon_e/\omega << 1$, fails only on the tail of the distribution where the number of particles [and, therefore, the contribution to the integral] is exponentially samll. We then therefore expand,

$$\frac{1}{\upsilon - (\omega/k)} = -\frac{k}{\omega}\frac{1}{1 - k\upsilon/\omega} = -\frac{k}{\omega}[1 + \frac{k\upsilon}{\omega} + (\frac{k\upsilon}{\omega})^2 + (\frac{k\upsilon}{\omega})^3 + ...] \tag{2}$$

$$\begin{aligned} 1 &= \frac{\omega_p^2}{k^2}\int d\upsilon \frac{\partial \hat{f}_0}{\partial \upsilon}(-\frac{k}{\omega})[1 + \frac{k\upsilon}{\omega} + (\frac{k\upsilon}{\omega})^2 + (\frac{k\upsilon}{\omega})^3 + ...] \\ &= \frac{\omega_p^2}{k^2}\int d\upsilon \frac{\partial \hat{f}_0}{\partial \upsilon}(-\frac{k}{\omega})[\frac{k\upsilon}{\omega} + (\frac{k\upsilon}{\omega})^3 + ...] \end{aligned}$$

where we have used the odd parity of, $\frac{\partial \hat{f}_0}{\partial \upsilon}$, in obtaining the last expression. Fially we integrate by parts and evaluted to give,

$$1 = \frac{\omega_p^2}{\omega^2} + 3\frac{\omega_p^2 k^2}{\omega^4}\bar{\upsilon}^2$$

or

$$\omega^2 = \omega_p^2 + \frac{\omega_p^2}{\omega^2}\frac{3KT_e}{m}k^2$$

The dispersion relation can be found by iteration, first letting, $\omega^2 = \omega_p^2$, and then evalutatin the correction term at, $\omega^2 = \omega_p^2$.

This yields,

$$\omega^2 = \omega_p^2(1 + \frac{3KT_e}{m}k^2)$$

which is the Bohm-Gross dispersion relation.

We have previously obtained this result from fluid theory assuming one-dimensional adiabatic compression ($\gamma = (D+2)/D = 3$) for an equation of state. The only proof of such an assumption, however, can come from kinetic theroy.

In deriving the Bohm-Gross dispersion relation, we have been somewhat glib about the mathematical expansion 2, since we have ignored extirely the singularity at $v = \omega/k$. Until we hace some method of resolving this potential contribution to the intergral, there is really no basis for the approximations that we were made. In essence, although this was not even motioned, we used the Vlasov prescription to resolve this singularity. Vlasov made a well-defined problem by replacing the singular funcition by its principal value, or, formally letting,

$$\frac{1}{v - (\omega/k)} \rightarrow P\frac{1}{v - (\omega/k)}$$

where the principal value operation, P, is defined according to,

$$\int_{-\infty}^{\infty} g(u)P\frac{1}{u-u_0}du = \lim_{\epsilon\rightarrow 0}\int_{-\infty}^{u_0-\epsilon} g(u)\frac{1}{u-u_0}du + \lim_{\epsilon\rightarrow 0}\int_{u_0+\epsilon}^{\infty} g(u)\frac{1}{u-u_0}du$$

Here g(u) is assumed to be analytic at, $u = u_0$. The limit then exists and is finite. One can easily see this by evaluating the (singular) contributions in the vicinity of, $u = u_0$, by letting $g(u) = g(u_0)$

$$\begin{aligned}
\int_{-\infty}^{\infty} g(u)P\frac{1}{u-u_0}du &\rightarrow g(u_0)\int P\frac{1}{u-u_0}du \\
&= g(u_0)[\lim_{\epsilon\rightarrow 0}\int^{u_0-\epsilon}\frac{1}{u-u_0}du + \lim_{\epsilon\rightarrow 0}\int_{u_0+\epsilon}\frac{1}{u-u_0}du] \\
&= g(u_0)\lim_{\epsilon\rightarrow 0}[\ln\epsilon - \ln\epsilon] = 0
\end{aligned}$$

The sigular contributions have opposite signs and cancel. The singularity can be resolved mathematically, by the application of the principal value. This resolution was tacitly assumed in our evalution of the dispersion relation above. It is Vlasov's solution.

Unfortunately, this resolution of the singularity is not uniqure. THere are infinitely many resolutions (all equally good mathematically) which would give different dispersion relations. The principal value prescription is completely ad hoc and without any physical basis. It would therefore be surprising if this particular choice was correct. In general, it is not. In erring, Vlasov missed one of the most important phenomenon in plasma physics - a damping arising from the proper resolution of this pole.

1.2 Brief Review of Integrals in the Complex Plane

1. *Cauch's Theorem:*

Let $f(z)$ be an analytic function over some region R of the complex plane, and let S be a simple closed curve whose interior is within R. Then

$$\oint_S f(z)dz = 0$$

An equivalent statement of the theorem, which is more useful in practice, is that a contour integral connecting two points may be deformed at will into any region of the complex plane where the integrand in analytic. As long as the end points remain fixed the value of the integral is unchanged.

2. *Cauchy's Residue Theorem:*

Let $f(z)$ be analytic and single-valued everywhere inside a closed contour C, except for a finite number of isolated singularites, then,

$$\oint_C f(z)dz = 2\pi i \sum_{poles\ z_i} \text{res } f(z_i)$$

where the contour, C, is taken in the counterclockwise direction.

For a simple pole at, $z = z_0$, the residue, res $f(z_i)$, is given by

$$\text{res } f(z_i) = \lim_{z \to z_0} (z - z_0) f(z_0)$$

1.3 Landau's Solution

The integral in 1 was treated properly by Landau. Since in practice ω is almost nerver real; waves are usually slightly damped by collisions or are amplified by some instablitiy mechanism and the velocity v is a real quanity, one might think that the singularity would be of now concern. Landau found that even though the singularity lies off the path of integration, its presence introduces an important modification to plamma wave dispersion reation - an effect not predicted by the fluid theory.

Consider an initial value problem in which the plasma is given a sinusodal perturbation, and therefore k is real. If the perturbation grows or decays, ω

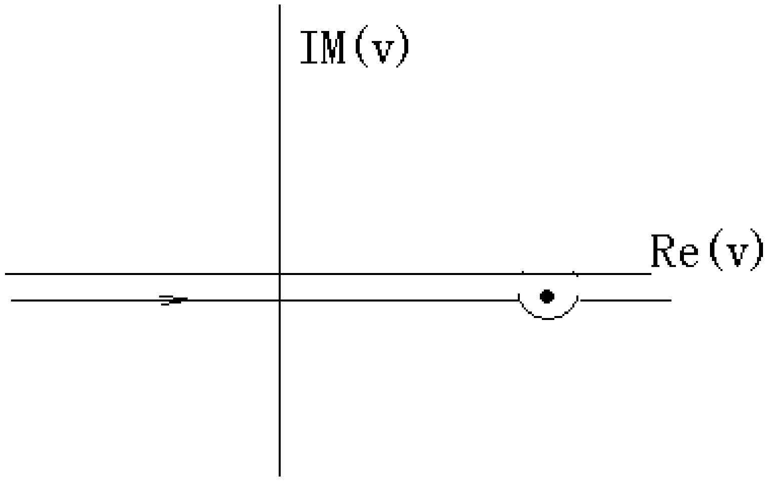


Figure 1:

will be complex. The integral in 1 must be treated as a contour integral in the complex v. Normally, one would evaluate the line integral along the real v axis by the residue theorem:

$$\int_{C_1} G dv + \int_{C_2} G dv = 2\pi i R(\omega/k)$$

Where G is the integrand, C_1 is the path along the real axis, C_2 is the semicircle at infinity, and $R(\omega/k)$ is the residue at ω/k. This works if the integral over C_2 vanishes. Unfortunately, this does not happen for a Maxwellian distribution, which contains the factor

$$\exp(-v^2/v_e^2)$$

This factor becomes large for $v \to \pm i\infty$, and the contribution from C_2 cannot be neglected. Landau showed that when the problem is properly treated as an initial value problem the correct contour to use is the curve C_1 passing below the singularity.

Although an exact analysis of this problem is complicated, we can obtain an an approximate dispersion reation for the case of large phase velocity and weak damping. In this case, the pole ω/k lies near the real v axis. The contour prescribed by Landau is then a straight line along the Re(v) axis with a semicircle around the pole.

In going around the pole, one obtains $2\pi i$ times half the residue there. Then

Eq 1 becomes

$$1 = \frac{\omega_p^2}{k^2}[P \int \frac{\partial \hat{f}_0/\partial \upsilon}{\upsilon - (\omega/k)} d\upsilon + i\pi \frac{\partial \hat{f}_0}{\partial \upsilon}|_{\upsilon=\omega/k}]$$

The first term of RHS is Vlasov's solution as we evalutated before to give the Bohm-Gross dispersion relation. We now return to the imaginary term (second term of RHS). In evaluating this small term, it will be sufficiently accurate to neglect the thermal correction to the real part of ω and let $\omega^2 \simeq \omega_p^2$. The principal value of the integral is approximately k^2/ω^2.

$$1 = \frac{\omega_p^2}{\omega^2} + i\pi \frac{\omega_p^2}{k^2} \frac{\partial \hat{f}_0}{\partial \upsilon}|_{\upsilon=\omega/k}$$

$$\omega^2 (1 - i\pi \frac{\omega_p^2}{k^2} \frac{\partial \hat{f}_0}{\partial \upsilon}|_{\upsilon=\omega/k}) = \omega_p^2$$

Treating the imaginary term as small,

$$\omega = \omega_p (1 + i\frac{\pi}{2} \frac{\omega_p^2}{k^2} \frac{\partial \hat{f}_0}{\partial \upsilon}|_{\upsilon=\omega/k})$$

For a Maxwellian distribution

$$\hat{f}_0 = (\pi \upsilon_e^2)^{-1/2} \exp(-\frac{\upsilon^2}{\upsilon_e^2})$$

$$\begin{aligned} \frac{\partial \hat{f}_0}{\partial \upsilon} &= (\pi \upsilon_e^2)^{-1/2} (\frac{-2\upsilon}{\upsilon_e^2}) \exp(-\frac{\upsilon^2}{\upsilon_e^2}) \\ &= -\frac{2\upsilon}{\sqrt{\pi}\upsilon_e^3} \exp(-\frac{\upsilon^2}{\upsilon_e^2}) \end{aligned}$$

We may approximate ω/k by ω_p/k in the coefficient, but in the exponent we must keep the thermal correction. The damping is then given by

$$\mathrm{Im}(\omega) \simeq -\frac{\pi}{2} \frac{\omega_p^2}{k^2} \frac{2\omega_p}{k\sqrt{\pi}\upsilon_e^3} \exp(-\frac{\omega^2}{k^2 \upsilon_e^2})$$

The difference between ω and ω_p may not be important in the outside but ought to be retained inside the exponential since

$$\frac{1}{\upsilon_e^2} \frac{\omega_p^2}{k^2} [1 + 3\frac{KT}{m} \frac{k^2}{\omega_p^2}] = \frac{\omega_p^2}{\upsilon_e^2 k^2} + \frac{3}{2}$$

so

$$\mathrm{Im}(\omega) = -\sqrt{\pi}\omega_p (\frac{\omega_p}{k\upsilon_e})^3 \exp(-\frac{\omega_p^2}{\upsilon_e^2 k^2}) \exp(-3/2)$$

or

$$\mathrm{Im}(\frac{\omega}{\omega_p}) = -0.22\sqrt{\pi}_p (\frac{\omega_p}{k\upsilon_e})^3 \exp(\frac{-1}{2k^2\lambda_D^2})$$

Since $\mathrm{Im}(\omega)$ is negative, there is a collisionless damping of plasma waves; this is called *Landau damping.*

1.4 Intuitive Physical Demonstration of Landau Damping

(Landau Damping without complex variables!)

We show by a direct calculation that net energy is transforred to electrons.

Suppose there exists a longitudianl wave

$$\mathbf{E} = E \cos(kz - \omega t)\hat{z}$$

Equations of motion of a particle

$$\begin{aligned}\frac{dv}{dt} &= \frac{q}{m} E \cos(kz - \omega t) \\ \frac{dz}{dt} &= v\end{aligned}$$

Solve these assuming E is small by perturbation expansion $v = v_0 + v_1 + ..., z = z_0(t) + z_1(t) + ...$

Zeroth order:

$$\frac{dv_0}{dt} = 0 \Rightarrow v_0 = const, z_0 = z_i + v_0 t$$

where z_i = const is the initial position.

First Order:

$$\begin{aligned}\frac{dv_1}{dt} &= \frac{q}{m} E \cos(kz_0 - \omega t) = \frac{q}{m} E \cos(k(z_i + v_0 t) - \omega t) \\ \frac{dz_1}{dt} &= v_1\end{aligned}$$

Integrate:

$$v_1 = \frac{qE}{m} \frac{\sin(kz_i + kv_0 t - \omega t)}{kv_0 - \omega} + const.$$

Take initial conditions to be $v_1|_{t=0} = 0$

Then

$$v_1 = \frac{qE}{m} \frac{\sin(kz_i + \Delta\omega t) - \sin(kz_i)}{\Delta\omega}$$

where $\Delta\omega \equiv kv_0 - \omega$

$$z_1 = \frac{qE}{m} [\frac{\cos kz_i - \cos(kz_i + \Delta\omega t)}{\Delta\omega^2} - t \frac{\sin(kz_i)}{\Delta\omega}$$

(using $z_1(0) = 0$)

2nd Order (needed to get energy right)

$$\begin{aligned}\frac{dv_2}{dt} &= \frac{qE}{m}\{\cos(kz_i + kv_0 t - \omega t + kz_1) \\ &\quad - \cos(kz_i + kv_0 t - \omega t)\} \\ &= \frac{qE}{m} kz_1\{-\sin(kz_i + \Delta\omega t)\} \quad (\mathrm{kz_1 << 1})\end{aligned}$$

Now the gain in kinetic energy of the particle is

$$\begin{aligned}\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 &= \frac{1}{2}m\{(v_0 + v_1 + v_2 + ...)^2 - v_0^2\} \\ &= \frac{1}{2}m\{2v_0v_1 + v_1^2 + 2v_0v_2 + higher\_order\}\end{aligned}$$

and the rate of increase of K.E. is

$$\frac{d}{dt}(\frac{1}{2}mv^2) = m(v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt})$$

We need to average this over space, i.e. over z_i. This will cancel any component that simply oscillates with z_i

$$< \frac{d}{dt}(\frac{1}{2}mv^2) >=< v_0\frac{dv_1}{dt} + v_1\frac{dv_1}{dt} + v_0\frac{dv_2}{dt} > m$$

$$< v_0\frac{dv_1}{dt} >= 0$$

$$\begin{aligned}&< \quad v_1\frac{dv_1}{dt} >=< \frac{q^2E^2}{m^2}[\frac{\sin(kz_i + \Delta\omega t) - \sin kz_i}{\Delta\omega}\cos(kz_i + \Delta\omega t)] > \\ &= \quad \frac{q^2E^2}{m^2} < \frac{\sin\Delta\omega t}{\Delta\omega}\cos^2(kz_i + \Delta\omega t) > \\ &= \quad \frac{q^2E^2}{m^2}\frac{1}{2}\frac{\sin\Delta\omega t}{\Delta\omega}\end{aligned}$$

$$\begin{aligned}&< \quad v_0\frac{dv_2}{dt} >= -\frac{q^2E^2}{m^2}kv_0 < (\frac{\cos kz_i - \cos(kz_i + \Delta\omega t)}{\Delta\omega} - t\frac{\sin kz_i}{\Delta\omega})\sin(kz_i + \Delta\omega t) > \\ &= \quad -\frac{q^2E^2}{m^2}kv_0 < (\frac{\sin\Delta\omega t}{\Delta\omega^2} - t\frac{\cos\Delta\omega t}{\Delta\omega})\sin^2(kz_i + \Delta\omega t) > \\ &= \quad \frac{q^2E^2}{m^2}\frac{kv_0}{2}[-\frac{\sin\Delta\omega t}{\Delta\omega^2} + t\frac{\cos\Delta\omega t}{\Delta\omega}]\end{aligned}$$

Hence

$$\begin{aligned}&< \quad \frac{d}{dt}(\frac{1}{2}mv^2) >= \frac{q^2E^2}{2m}[\frac{\sin\Delta\omega t}{\Delta\omega} - kv_0\frac{\sin\Delta\omega t}{\Delta\omega^2} + kv_0 t\frac{\cos\Delta\omega t}{\Delta\omega}] \\ &= \quad \frac{q^2E^2}{2m}[-\frac{\omega\sin\Delta\omega t}{\Delta\omega^2} + \frac{\omega t}{\Delta\omega}\cos\Delta\omega t + t\cos\Delta\omega t]\end{aligned}$$

This is the space-averaged power into particles of a specific velocity v_0. We need to integrate over the distribution funcition. A trick identity helps:

$$
\begin{aligned}
-\frac{\omega \sin \Delta\omega t}{\Delta\omega^2} + \frac{\omega t}{\Delta\omega}\cos\Delta\omega t + t\cos\Delta\omega t &= \frac{\partial}{\partial \Delta\omega}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right) \\
&= \frac{1}{k}\frac{\partial}{\partial v_0}\left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right)
\end{aligned}
$$

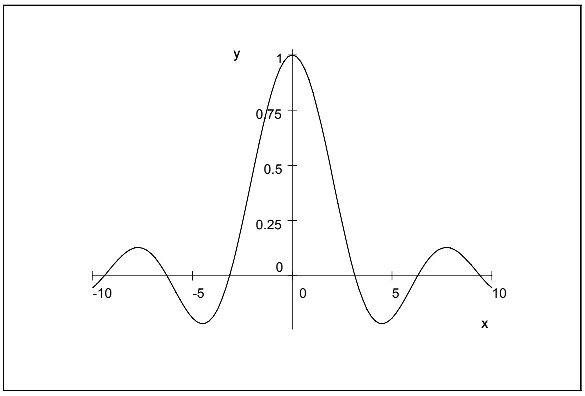
Hence power per unit volume is

$$
\begin{aligned}
P &= \int < \frac{d}{dt}(\frac{1}{2}mv^2) > f(v_0)dv_0 \\
&= -\frac{q^2E^2}{2mk}\int \left(\frac{\omega\sin\Delta\omega t}{\Delta\omega} + \sin\Delta\omega t\right)\frac{\partial f}{\partial v_0}dv_0
\end{aligned}
$$

As t becomes large $\sin\Delta\omega t = \sin(kv_0-\omega)t$ becomes a rapidly oscillating function of v_0. Hecne second term of integrand contributes negligibly and the first term $\propto \frac{\omega\sin\Delta\omega t}{\Delta\omega} = \frac{\sin\Delta\omega t}{\Delta\omega t}\omega t$

becomes a highly localized, delta-function-like quantity.

$$\frac{\sin x}{x}$$



That enables the rest of the integrand to be evaluated just where $\Delta\omega = 0$ $(i.e.\ kv_0 - \omega = 0)$

So

$$P = -\frac{q^2E^2}{2mk}\frac{\omega}{k}\frac{\partial f}{\partial v}|_{\omega/k}\int \frac{\sin x}{x}dx$$

$x = \Delta\omega t = (kv_0 - \omega)t$

and $\int \frac{\sin x}{x}dx = \pi$ so

$$P = -E^2\frac{\pi q^2\omega}{2mk^2}\frac{\partial f_0}{\partial v}|_{\omega/k}$$

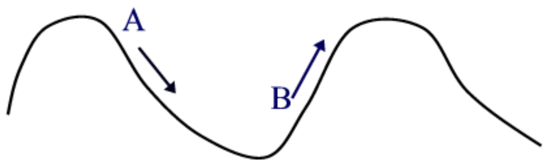


Figure 2:

We have shown that there is a net transfer of energy to particles at the resonant velocity ω/k from the wave. (Positive if $\frac{\partial f_0}{\partial v}|$ is negative)

<u>Physical Picture</u>
$\Delta\omega$ is the frequency in the particles (unperturbed) frame of reference, or equivalently it is kv_0' where v_0' is particle speed in wave frame of reference. The latter is easier to deal with. $\Delta\omega t = kv_0' t$ is the phase the particle travels in time t. We found that the energy gain was of the form

$$\int \frac{\sin \Delta\omega t}{\Delta\omega t} d(\Delta\omega t)$$

This integrand becomes small (and oscillatory) for $\Delta\omega t >> 1$. Physically, this means that if particle moves through many wavelengths its energy gain is small. Dorminant contribution is form $\Delta\omega t << \pi$. There are particles that move throught less than 1/2 wavelength during the period under consideration. These are the resonant particles.

Particles moving slightly faster thatn wave are slowed down. This is a second-order effect. Some particles of this v_0 group are being accelerated (A) some slowed (B). Becauses A's are then going faster, they speed less time in the 'down' region. B's are slowed; they spend more time in up region. Net effect: tendency for particle to move its speed toward that of wave.

Particle moving slightly slower than wave are speeded up. (some argument). But this is only true for particles that have 'caught the wave'.

Summary: Resonant particles' velocity is drawn toward the wave phase velocity.

Is there net energy when we average over both slower and faster particles? Depends which type has most.

Picutre: Net particle energy gain, Wave Damped
Net particle energy loss, Wave Amplified

Our complex variables wave treatment and our direct particle energy calculation give consistent answers. To show this we need to show energy conservation.

Energy density of wave:

$$\overline{W} = \frac{1}{2}[\frac{1}{2}\epsilon_0|E^2| + n\frac{1}{2}m|v^2|]$$

1/2: from average of $\sin^2$
$\frac{1}{2}\epsilon_0|E^2|$: electrostatic
$n\frac{1}{2}m|v^2|$: Particle Kinetic

Magnetic wave energy zero for a longitudinal wave, we showed in Cold plasma treatment

$$v \simeq \frac{qE}{-i\omega m}$$

Hence

$$\overline{W} \simeq \frac{1}{2}\frac{\epsilon_0 E^2}{2}[1 + \frac{\omega_p^2}{\omega^2}] \quad \text{(again electrons only)}$$

When the wave is damped, it has imaginary part of ω, ω_i and

$$\frac{d\overline{W}}{dt} = \overline{W}\frac{1}{E^2}\frac{dE^2}{dt} = 2\omega_i\overline{W}$$

Conservation of energy requires that this equal minus the particle energy gain rate, P, Hence

$$\omega_i = \frac{-P}{2W} = \frac{E^2 \frac{\pi q^2 \omega}{2mk^2}\frac{\partial f_0}{\partial v}|_{\omega/k}}{\frac{\epsilon_0 E^2}{2}[1 + \frac{\omega_p^2}{\omega^2}]} = \omega_p^2 \frac{\pi}{2}\frac{\omega}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\omega/k} \times \frac{2}{1 + \frac{\omega_p^2}{\omega^2}}$$

so for waves such that $\omega \simeq \omega_p$, which is the dispersion relation to lowest order, we get

$$\omega_i = \omega_p^2 \frac{\pi}{2}\frac{\omega_r}{k^2}\frac{1}{n}\frac{\partial f_0}{\partial v}|_{\omega_r/k}$$

This exactly agrees with the damping calculatied from the complex dispersion relation using the Vlasov equation.

This is the Landau damping calculation for longitudinal waves in a (magnetic) field-free plasma. Strictly, just for elecron plasma waves.