IOP PUBLISHING PLASMA PHYSICS AND CONTROLLED FUSION

Plasma Phys. Control. Fusion **50** (2008) 105002 (13pp) doi:10.1088/0741-3335/50/10/105002

Energetic particles modeled by a generalized relativistic kappa-type distribution function in plasmas

Fuliang Xiao[1,2], **Liangxu Chen**[1] **and Jiangfan Li**[1]

[1] School of Physics and Electronic Sciences, Changsha University of Science and Technology, Changsha 410076, People's Republic of China
[2] State Key Laboratory of Space Weather, PO Box 8701, Beijing 100080, People's Republic of China

Received 28 May 2008, in final form 24 July 2008
Published 12 August 2008
Online at stacks.iop.org/PPCF/50/105002

Abstract

Energetic particles found in planetary magnetospheres and other plasmas often display a power law and an anisotropy (including loss cone and temperature anisotropy). In a recent study, a full relativistic kappa-loss cone (KLC) distribution $f^{\kappa L}$ is initially introduced to model energetic particles, but $f^{\kappa L}$ is only associated with loss cone anisotropy. We extend this previous study and develop a generalized relativistic kappa-type (KT) distribution $f^{\kappa T}$ which incorporates either temperature anisotropy or both loss cone and temperature anisotropy. We carry out numerical calculations for a direct comparison between the new KT distribution, the previous KLC distribution and the kappa distribution f^{κ} , respectively. We find that (a) analogous to $f^{\kappa L}$, $f^{\kappa T}$ satisfies the power law not only at lower energies but also at relativistic energies; (b) analogous to f^{κ}, $f^{\kappa T}$ contains either temperature anisotropy or both loss cone and temperature anisotropy; (c) the regular kappa distribution is found to decrease faster than the KT distribution with kinetic energy E_k especially when θ^2 increases (where θ^2 is the thermal characteristic parameter), e.g. $f^{\kappa}/f^{\kappa T} \lesssim 10^{-2}$ for $E_k \gtrsim 2.0$ MeV and $\theta^2 \gtrsim 0.3$; (d) no big difference occurs between both KT and kappa distributions through energies up to ∼500 keV for $\theta^2 \lesssim 0.03$ and (e) the three distributions show different anisotropy behaviors even for the same overall anisotropy. The results suggest that the new generalized KT distribution may be applied in space plasmas and other plasmas including laboratory machines where highly energetic particles exist.

1. Introduction

Energetic particles (with energies of hundreds of keV or above) have been found to play a critical role in the dynamics of space plasmas or other plasmas by wave–particle interaction. Similarly to the big effect of the solar wind on the Earth's magnetosphere [1–3], solar energetic particles associated with the solar flare, coronal and interplanetary shocks have a great impact on the space weather [4]. Energetic ('killer') electrons in the outer radiation belt of the Earth have been found to yield the failure and malfunctions of geostationary spacecraft [5]. Two fundamental mechanisms have been proposed to account for acceleration of those energetic electrons: inward radial diffusion through drift resonance with enhanced ULF waves [6–10] and cyclotron wave–particle interaction through Doppler shift resonance [11–17]. Hence, specifying and even forecasting fluxes (or distribution) of those energetic electrons during and following geomagnetic storms constitute important facets of space weather science. One crucial way is to find an appropriate distribution function to model energetic particle behavior.

Electromagnetic wave instability, which primarily governs time scales for acceleration, is found to be associated with the behavior of particle distributions or fluxes [18–21] since plasma turbulence can interact with both thermal and superthermal tails of the particle velocity [22]. The typically hot, tenuous and collisionless space plasma generally has a power-law energy tail and an anisotropy (including loss cone and temperature anisotropy) and can be basically well modeled by a generalized Lorentzian (kappa) distribution [23–25]. The kappa distribution has been extensively adopted in numerous previous works [26–29] to model highly energetic particle behavior. Various observations, including geostationary orbit data [30], electron measurements from LANL satellites and GOES 10 [31] and energy spectra of solar events [32], have demonstrated that energetic particles can be well characterized by a power-law spectrum. In previous works [33, 34], Summers *et al* have presented a theoretical evaluation of the modified plasma dispersion function and found general properties of ion waves and Langmuir waves for plasmas in which both electrons and ions have kappa distribution. Furthermore, Summers *et al* have derived the general properties of the dielectric tensor of the kappa distribution function [35]. Meanwhile, Mace [36] has derived a dielectric tensor for the kappa distribution which extends the results of [35] and yields a form which permits further progress in analytical studies of wave propagation perpendicular to the magnetic field [37,38]. A previous work [39] has also studied generalized Langmuir waves in a magnetized plasma with a Maxwellian–Lorentzian distribution.

Recently, Xiao [40] developed a fully relativistic kappa-loss cone (KLC) distribution to model highly energetic particles in plasmas where magnetic mirror geometries occur. The KLC distribution is found to combine these features of the well-known kappa type (KT) and loss cone type and follow the power law at both lower energies and relativistic energies. Furthermore, Xiao *et al* [41, 42] adopt the KLC distribution (without consideration of anisotropy) to fit solar energetic particle spectra observed by the IMP 8 and Helios 1 and 2 spacecraft and the energetic electron spectrum observed by the SOPA instrument on board the 1989-046 and LANL-01A satellites at the geosynchronous orbit. Xiao *et al* find that the KLC distribution fits well with the observed data during different universal times both in lower and higher energies. However, the KLC distribution is only associated with loss cone anisotropy. Since energetic particles found in planetary magnetospheres and other plasmas often display both loss cone and temperature anisotropy, hence in order to model energetic particles in a more physically realistic way, a generalized relativistic distribution which incorporates both loss cone and temperature anisotropy needs to be developed. This is the primary goal of this study.

2. General development

2.1. Previous distributions

First, we present a brief description of the previous relativistic KLC distribution $f^{\kappa L}$ which is associated with only loss cone anisotropy and the kappa distribution, respectively.

Analogous to the well-known DGH loss cone type distribution [43], the previous relativistic KLC distribution $f^{\kappa L}$ has been introduced by Xiao [40]:

$$f^{\kappa L}(p,\alpha) = \frac{1}{2\pi^{3/2}}\frac{\Gamma((q+3)/2)}{\Gamma((q+2)/2)}\frac{1}{I_{\kappa L}^2}\left[1+\frac{\sqrt{1+p^2}-1}{\kappa\theta^2}\right]^{-(\kappa+1)}\sin^q\alpha, \tag{1}$$

where Γ is the gamma function, $p = (p_\perp^2 + p_\parallel^2)^{1/2}$ is the total momentum of a particle normalized by mc, c is the speed of light and m is the rest mass of particle; $p_\parallel$ and $p_\perp$ both scaled by mc are the particle momenta in directions along and perpendicular to the magnetic field, $\alpha = \arctan(p_\perp/p_\parallel)$ is the particle pitch angle, κ is the spectral index, q represents the pitch-angle anisotropy and θ^2 is the thermal characteristic parameter scaled by mc^2 and $I_{\kappa L}^2$ is the normalized constant given by

$$\begin{aligned} I_{\kappa L}^2 = \frac{8B(3/2,\kappa-2)}{2\kappa-1}&\{3F(\kappa+1,5/2;\kappa+1/2;1-2/\kappa\theta^2) \\ &+(\kappa-2)F(\kappa+1,3/2;\kappa+1/2;1-2/\kappa\theta^2)\}, \end{aligned} \tag{2}$$

where B and F are beta and hypergeometric functions respectively.

The well-known non-relativistic generalized Lorentz (kappa) distribution associated with loss cone and temperature anisotropy in the momentum space can be written [44] as

$$\begin{aligned} f^{\kappa}(p_\parallel, p_\perp) = &\frac{\Gamma(\kappa+l+1)}{\pi^{3/2}\theta_\perp^2\theta_\parallel\kappa^{(l+3/2)}\Gamma(l+1)\Gamma(\kappa-1/2)} \\ &\left(\frac{p_\perp}{\theta_\perp}\right)^{2l}\left[1+\frac{p_\parallel^2}{\kappa\theta_\parallel^2}+\frac{p_\perp^2}{\kappa\theta_\perp^2}\right]^{-(\kappa+l+1)}, \end{aligned} \tag{3}$$

here l, similar to the index 'q' in a typical loss cone distribution (i.e. $\propto \sin^q\alpha$), represents the pitch angle anisotropy associated with a measure of the angular size of the loss cone region; $\theta_\parallel^2$ and $\theta_\perp^2$ are thermal characteristic parameters (scaled by mc^2) associated with temperature anisotropy. Here we substitute the momentum for the velocity in the distribution function because it is more appropriate when the velocity approaches the speed of light. Previous works have used a non-relativistic distribution function similar in form to relation (3) in studies of electromagnetic R-mode and L-mode waves [45] and in evaluation of whistler instability in the Earth's foreshock [46]. Xiao *et al* [41,42] have demonstrated that a distribution analogous to relation (3) fits well with observational data at relatively lower energies of electrons but displays deviations at higher energies, typically above hundreds of keV.

2.2. New KT distribution

In order to model energetic particles in plasmas by a more physically realistic way, we assume a new generalized relativistic KT distribution to take the form

$$f^{\kappa T}(p_\parallel, p_\perp) = C_{\kappa T}\left(\frac{p_\perp}{\beta_\perp}\right)^{2l}\left[1+\frac{\sqrt{1+p_\parallel^2/\beta_\parallel^2+p_\perp^2/\beta_\perp^2}-1}{\kappa\theta^2}\right]^{-(\kappa+l+1)}, \tag{4}$$

where $\beta_\parallel^2$ and $\beta_\perp^2$ are introduced dimensionless characteristic parameters associated with temperature anisotropy, $C_{\kappa T}$ is the normalized coefficient of the distribution function defined by $\int \mathrm{d}^3p f^{\kappa T}(p_\parallel, p_\perp) = 1$ and $\mathrm{d}^3p = 2\pi p_\perp\,\mathrm{d}p_\perp\,\mathrm{d}p_\parallel = 2\pi p^2\sin\alpha\,\mathrm{d}p\,\mathrm{d}\alpha$. One of our following jobs is to define $C_{\kappa T}$.

By taking the following variable transformations: $p_\parallel = \beta_\parallel \hat{p}_\parallel$ and $p_\perp = \beta_\perp \hat{p}_\perp$, the KT distribution (4) becomes

$$f^{\kappa T}(\hat{p}_\parallel, \hat{p}_\perp) = C_{\kappa T}\hat{p}_\perp^{2l}\left[1 + \frac{\sqrt{1+\hat{p}_\parallel^2+\hat{p}_\perp^2}-1}{\kappa\theta^2}\right]^{-(\kappa+l+1)}. \tag{5}$$

A further assumption of $\hat{p}_\parallel = \hat{p}\cos\hat{\alpha}$ and $\hat{p}_\perp = \hat{p}\sin\hat{\alpha}$ leads to the result

$$f^{\kappa T}(\hat{p}, \hat{\alpha}) = C_{\kappa T}\hat{p}^{2l}\left[1 + \frac{\sqrt{1+\hat{p}^2}-1}{\kappa\theta^2}\right]^{-(\kappa+l+1)}\sin^{2l}\hat{\alpha} = C_{\kappa T}\hat{p}^{2l}f_1(\hat{p})\sin^{2l}\hat{\alpha}. \tag{6}$$

Assuming $\sqrt{1+\hat{p}^2}-1 = x$, we obtain

$$\hat{p}^2 = x(x+2), \qquad \mathrm{d}\hat{p} = \frac{x+1}{\sqrt{x(x+2)}}\,\mathrm{d}x. \tag{7}$$

It is useful to start with the evaluation of integrals involved with $\hat{p}^n$ $(n > 0)$ to obtain more general results:

$$\int_0^\infty \mathrm{d}\hat{p}\,\hat{p}^n f_1(\hat{p}) = \int_0^\infty \mathrm{d}x\,\frac{[x(x+2)]^{\frac{n-1}{2}}(x+1)}{[1+x/\kappa\theta^2]^{\kappa+l+1}} = I^n_{\kappa T}. \tag{8}$$

Following a calculation similar to that used in [40], we obtain the relation

$$I^n_{\kappa T} = \frac{2^{n+1}B(\frac{n+1}{2}, \kappa+l-n)}{2\kappa+2l-n+1}\left\{(n+1)F\left(\kappa+l+1; \frac{n+3}{2}; \kappa+l-\frac{n}{2}+\frac{3}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right.$$
$$\left. + (\kappa+l-n)F\left(\kappa+l+1, \frac{n+1}{2}; \kappa+l-\frac{n}{2}+\frac{3}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right\}. \tag{9}$$

In the case of $n = 2l+2$, equation (9) returns to

$$I^{2l+2}_{\kappa T} = \frac{2^{2l+3}B(\frac{2l+3}{2}, \kappa-l-2)}{2\kappa-1}\left\{(2l+3)F\left(\kappa+l+1; \frac{2l+5}{2}; \kappa+\frac{1}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right.$$
$$\left. + (\kappa-l-2)F\left(\kappa+l+1; \frac{2l+3}{2}; \kappa+\frac{1}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right\}. \tag{10}$$

In the case of $n = 2l+4$, equation (9) reduces to

$$I^{2l+4}_{\kappa T} = \frac{2^{2l+5}B(\frac{2l+5}{2}, \kappa-l-4)}{2\kappa-3}\left\{(2l+5)F\left(\kappa+l+1, \frac{2l+7}{2}; \kappa-\frac{1}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right.$$
$$\left. + (\kappa-l-4)F\left(\kappa+l+1, \frac{2l+5}{2}; \kappa-\frac{1}{2}; 1-\frac{2}{\kappa\theta^2}\right)\right\}. \tag{11}$$

Using the normalization condition: $\int \mathrm{d}^3 p f(p_\parallel, p_\perp) = 1$, it is straightforward but a little tedious to show that

$$C_{\kappa T} = \frac{1}{2\pi^{3/2}}\frac{1}{\beta_\perp^2\beta_\parallel}\frac{\Gamma(l+3/2)}{\Gamma(l+1)}\frac{1}{I^{2l+2}_{\kappa T}}. \tag{12}$$

Substituting (12) into (4), the new KT distribution finally becomes

$$f^{\kappa T}(p_\parallel, p_\perp) = \frac{1}{2\pi^{3/2}}\frac{1}{\beta_\perp^2\beta_\parallel}\frac{\Gamma(l+3/2)}{\Gamma(l+1)}\frac{1}{I^{2l+2}_{\kappa T}}\left(\frac{p_\perp}{\beta_\perp}\right)^{2l}$$
$$\left[1 + \frac{\sqrt{1+p_\parallel^2/\beta_\parallel^2+p_\perp^2/\beta_\perp^2}-1}{\kappa\theta^2}\right]^{-(\kappa+l+1)}. \tag{13}$$

$f^{\kappa T}(p_\parallel, p_\perp)$ represented by (13) is a generalized relativistic KT distribution function in scaled variables and constitutes the main result in this study. The term $I_{\kappa T}^{2l+2}$ associated with the spectral index κ, the thermal characteristic parameter θ^2 and the loss cone index l is evaluated by equation (10).

We shall analyze some features of $f^{\kappa T}(p_\parallel, p_\perp)$ in the non-relativistic limit $\theta^2 \ll 1$ and the relativistic limit $\theta^2 \gg 1$, respectively, as follows.

(a) If $\theta^2 \ll 1$, leading to $\sqrt{1 + p_\parallel^2/\beta_\parallel^2 + p_\perp^2/\beta_\perp^2} - 1 \approx p_\parallel^2/2\beta_\parallel^2 + p_\perp^2/2\beta_\perp^2$. Then $f^{\kappa T} \propto [1+p_\parallel^2/(2\beta_\parallel^2\kappa\theta^2)+p_\perp^2/(2\beta_\perp^2\kappa\theta^2)]^{-(\kappa+l+1)}$, implying that $f^{\kappa T}$ reduces to the non-relativistic kappa distribution.

(b) If $\theta^2 \gg 1$, $\sqrt{1 + p_\parallel^2/\beta_\parallel^2 + p_\perp^2/\beta_\perp^2} - 1 \approx p/\beta$ (here β denotes a parameter scaling with $\beta_\parallel$ or $\beta_\perp$), i.e. $f^{\kappa T} \propto [1 + p/(\kappa\beta\theta^2)]^{-(\kappa+l+1)}$, indicating that the distribution follows a power law at relativistic energies.

(c) If $\kappa \to \infty$, $f^{\kappa T} \propto \exp[-\sqrt{1 + p_\parallel^2/\beta_\parallel^2 + p_\perp^2/\beta_\perp^2}/\theta^2]$, indicating that $f^{\kappa T}(p_\parallel, p_\perp)$ incorporates features of the standard relativistic Maxwellian distribution function (e.g. [47, 48]). Similarly, the energy part recovers to the usual non-relativistic Maxwellian if $\theta^2 \ll 1$ and spreads at relativistic energy ($\sim \exp[-p/\beta\theta^2]$) if $\theta^2 \gg 1$, consistent with previous results [47].

2.3. *Equilibrium relativistic Vlasov equation*

Here we shall present a concise calculation whether this new KT distribution (4) or (13) obeys the equilibrium relativistic Vlasov equation.

The standard Vlasov equation in both $\boldsymbol{r}$ and $\boldsymbol{v}(= v_x\hat{x} + v_y\hat{y} + v_z\hat{z})$ space for particles with mass m and charge q under an electric field $\boldsymbol{E}_0$ and an ambient magnetic field $\boldsymbol{B}_0$ can be written as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot (f\boldsymbol{v}) + \frac{\partial}{\partial \boldsymbol{v}} \cdot (f\boldsymbol{a}) = 0, \tag{14}$$

where $\boldsymbol{a}$ is the acceleration of the particles in the volume element, $\hat{x}$, $\hat{y}$ and $\hat{z}$ denote the unit vector of the coordinate system (x, y, z) in which the ambient magnetic field $\boldsymbol{B}_0$ lies in the z-direction, i.e. $\boldsymbol{B}_0 = B_0\hat{z}$. Since $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables in (14) we may bring $\boldsymbol{v}$ outside the differential operator. For an electromagnetic force: $\boldsymbol{a} = q/m(\boldsymbol{E}_0 + \boldsymbol{v} \times \boldsymbol{B}_0)$, then $\nabla_v \cdot \boldsymbol{a} = 0$, we can further obtain

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{a} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{15}$$

In the equilibrium state, the new KT distribution $f^{\kappa T}$ is independent of time t or space $\boldsymbol{r}$, then substitution of $f^{\kappa T}$ (13) into (15) leads to

$$\boldsymbol{a} \cdot \frac{\partial f^{\kappa T}}{\partial \boldsymbol{v}} = 0. \tag{16}$$

In the case of a mirror field geometry, it is reasonable to assume that the electric field vanishes ($\boldsymbol{E}_0 = 0$) in the equilibrium state. Otherwise, the positive and negative charge particles move in the opposite direction under the drive of the electric field $\boldsymbol{E}_0$, accumulating and forming a electric field in the anti-direction of $\boldsymbol{E}_0$ and finally canceling the electric field $\boldsymbol{E}_0$. Hence, we can further have

$$\boldsymbol{a} = \frac{q}{m}\boldsymbol{v} \times \boldsymbol{B}_0 = \frac{qB_0}{m}(v_y\hat{x} - v_x\hat{y}). \tag{17}$$

Equation (17) indicates that $\boldsymbol{a}$ is perpendicular to $\boldsymbol{v}$. Since we further have:

$$\frac{\partial f^{\kappa T}}{\partial \boldsymbol{v}} = \frac{\partial f^{\kappa T}}{\partial v_x}\hat{x} + \frac{\partial f^{\kappa T}}{\partial v_y}\hat{y} + \frac{\partial f^{\kappa T}}{\partial v_z}\hat{z}, \tag{18}$$

then from (16)–(18) we obtain

$$\frac{qB_0}{m}\left(v_y\frac{\partial f^{\kappa T}}{\partial v_x} - v_x\frac{\partial f^{\kappa T}}{\partial v_y}\right) = 0. \tag{19}$$

It is easily verified that the most general solution for (19) is

$$f^{\kappa T} = f^{\kappa T}(p_\perp, p_\parallel) \quad \text{or} \quad f^{\kappa T} = f^{\kappa T}(v_\perp, v_\parallel). \tag{20}$$

Therefore the new KT (13) distribution $f^{\kappa T}$ satisfies (16) and correspondingly obeys the equilibrium relativistic Vlasov equation.

Similarly, it is straightforward to show that the new KT distribution $f^{\kappa T}$ also satisfies the equilibrium relativistic Vlasov equation in both $\boldsymbol{r}$ and $\boldsymbol{p}$ space:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\cdot(f\boldsymbol{p}) + \frac{\partial}{\partial \boldsymbol{p}}\cdot(f\dot{\boldsymbol{p}}) = 0, \tag{21}$$

where $\dot{\boldsymbol{p}}$ is the change rate of the particles' momentum in the volume element.

2.4. Anisotropy and mean kinetic energy

In the following we shall evaluate the mean kinetic energy and anisotropy of energetic particles modeled by $f^{\kappa T}(p_\parallel, p_\perp)$. We adopt the following standard definitions for parallel and perpendicular mean square momenta [44], namely,

$$P_\parallel = \int p_\parallel^2 f(p_\parallel, p_\perp)\,\mathrm{d}^3p, \qquad P_\perp = \int p_\perp^2 f(p_\parallel, p_\perp)\,\mathrm{d}^3p. \tag{22}$$

Relation (22) may be directly associated with temperature in the non-relativistic limit ($T_\parallel \Rightarrow mc^2P_\parallel$; $T_\perp \Rightarrow mc^2P_\perp/2$). Analogous to the corresponding definition ($A = T_\perp/T_\parallel - 1$) in a non-relativistic theory, we therefore use (22) as a definition of 'temperature' of $T_\parallel$ and $T_\perp$ as in [44]. The thermal anisotropy may be defined as

$$A = \frac{P_\perp}{2P_\parallel} - 1. \tag{23}$$

The anisotropy $A^{\kappa L}$ of energetic particles for a KLC distribution can be written [40] as

$$A^{\kappa L} = \frac{P_\perp}{2P_\parallel} - 1 = \frac{q+2}{2} - 1 = \frac{q}{2} \tag{24}$$

with

$$P_\perp = \frac{I_{\kappa L}^4}{I_{\kappa L}^2}\frac{q+2}{q+3}, \qquad P_\parallel = \frac{I_{\kappa L}^4}{I_{\kappa L}^2}\frac{1}{q+3}, \tag{25}$$

where $I_{\kappa L}^2 = I_{\kappa T}^2(l=0)$, $I_{\kappa L}^4 = I_{\kappa T}^4(l=0)$. This suggests that both the KLC distribution and a typical loss cone distribution (namely, $\propto \sin^q\alpha$) have the same anisotropy parameter.

The anisotropy A^κ of energetic particles for the kappa distribution is given by

$$A^\kappa = \frac{P_\perp}{2P_\parallel} - 1 = \frac{(l+1)\theta_\perp^2}{\theta_\parallel^2} - 1 \tag{26}$$

with

$$P_\perp = \frac{2\kappa\theta_\perp^2(l+1)}{2\kappa-3}, \qquad P_\parallel = \frac{\kappa\theta_\parallel^2}{2\kappa-3}. \tag{27}$$

For the new KT distribution we have

$$P_\perp = \int p_\perp^2 f^{\kappa T}(p_\parallel, p_\perp)\,\mathrm{d}^3 p, \qquad P_\parallel = \int p_\parallel^2 f^{\kappa T}(p_\parallel, p_\perp)\,\mathrm{d}^3 p. \tag{28}$$

Using (10), (11) and (13), we further obtain

$$P_\perp = \frac{2\beta_\perp^2(l+1)}{2l+3}\frac{I_{\kappa T}^{2l+4}}{I_{\kappa T}^{2l+2}}, \qquad P_\parallel = \frac{\beta_\parallel^2}{2l+3}\frac{I_{\kappa T}^{2l+4}}{I_{\kappa T}^{2l+2}}. \tag{29}$$

Then the anisotropy $A^{\kappa T}$ of energetic particles becomes

$$A^{\kappa T} = \frac{P_\perp}{2P_\parallel} - 1 = \frac{(l+1)\beta_\perp^2}{\beta_\parallel^2} - 1. \tag{30}$$

This results imply that analogous to a kappa distribution, the overall anisotropy of the new KT distribution incorporates both loss cone and temperature anisotropy.

In the case of a relativistic limit, it is valuable to use the mean kinetic energy to evaluate the energy distribution of particles. For each distribution above, the scaled mean kinetic energy $\overline{E_k}$ can be obtained by

$$\overline{E_k} = \int \mathrm{d}^3 p(\sqrt{1+p^2} - 1) f(p_\parallel, p_\perp), \tag{31}$$

which may be evaluated by the standard numerical technique. The anisotropy and the mean kinetic energy behavior for the KT and kappa distribution functions above will be evaluated in detail in the following section.

3. Numerical results

In the following we will present calculations of both energy behavior and anisotropy behavior of energetic electrons modeled by the KT distribution in a direct comparison with the KLC and kappa distributions.

Since in this study, we focus on the temperature anisotropy, for the energy behavior, we shall not consider loss cone anisotropy and assume $q = 0$ in $f^{\kappa L}(p, \alpha)$; l=0 in both $f^\kappa(p_\parallel, p_\perp)$ and $f^{\kappa T}(p_\parallel, p_\perp)$. Then equations (1), (3) and (13) can be simplified, respectively, as

$$f^{\kappa L}(p, \alpha) = \frac{1}{4\pi I_{\kappa L}^2}\left[1 + \frac{\sqrt{1+p^2} - 1}{\kappa\theta^2}\right]^{-(\kappa+1)}, \tag{32}$$

$$f^\kappa(p_\parallel, p_\perp) = \frac{\Gamma(\kappa+1)}{\pi^{3/2}\theta_\perp^2\theta_\parallel\kappa^{3/2}\Gamma(\kappa - 1/2)}\left[1 + \frac{p_\parallel^2}{\kappa\theta_\parallel^2} + \frac{p_\perp^2}{\kappa\theta_\perp^2}\right]^{-(\kappa+1)} \tag{33}$$

and

$$f^{\kappa T}(p_\parallel, p_\perp) = \frac{1}{4\pi I_{\kappa T}^2}\frac{1}{\beta_\perp^2\beta_\parallel}\left[1 + \frac{\sqrt{1 + p_\parallel^2/\beta_\parallel^2 + p_\perp^2/\beta_\perp^2} - 1}{\kappa\theta^2}\right]^{-(\kappa+1)}. \tag{34}$$

Since $\sqrt{1+p^2} - 1 \approx p^2/2$ in (1) and (13) in the non-relativistic limit, without loss of generality, we assume $\beta_\parallel^2 = 1$ and $\theta_\parallel^2 = 2\theta^2$ in the following calculation in order to make a consistent and direct comparison.

In the case of isotropy: $A = 0$ (or $\beta_\parallel^2 = \beta_\perp^2$), since $I_{\kappa L}^2 = I_{\kappa T}^2(l = 0)$, relations (32) and (34) are the same, indicating that the new KT and KLC distributions have the same energy behavior. In particular, relation (32) or (34) is found to fit well with data of the solar energetic

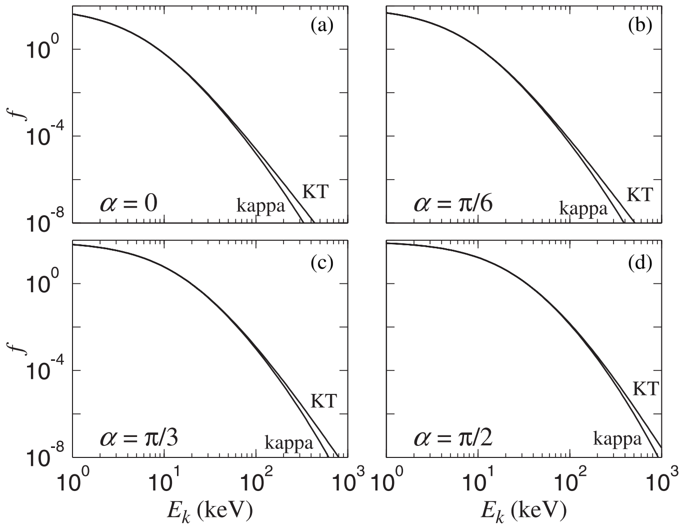

Figure 1. Curves of the KT (34) and kappa (33) distributions for $A = 3$, $\kappa = 4.5$, $\theta^2 = 0.003$ and different indicated values of pitch angle α. Since θ^2 is scaled by $m_e c^2$ (i.e. ~500 keV), θ^2 approximately corresponds to 1.5 keV.

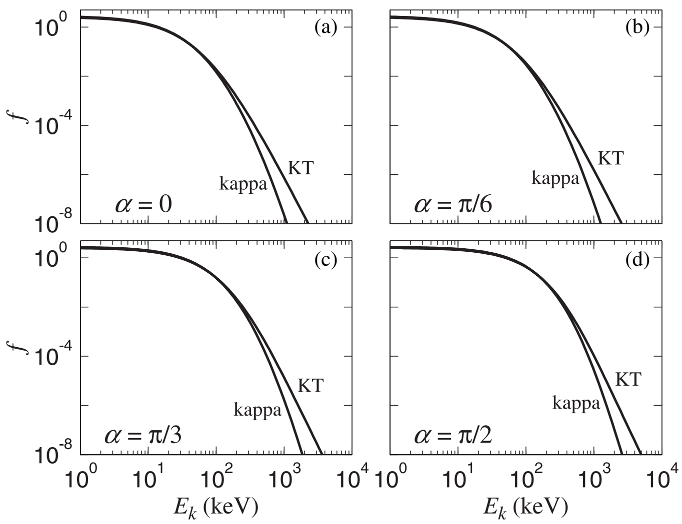

Figure 2. Same as figure 1 except $\theta^2 = 0.03$ (~15 keV).

particle and the geostationary orbital energetic electrons spectrum at both lower and relativistic energies [41, 42]. Since KLC is not associated with temperature anisotropy, we therefore shall not discuss the energy behavior of KLC distribution and refer the reader to [40] for full details.

In figures 1–4, we present curves of the new KT distribution (34) and the regular kappa distribution (33) for $\kappa = 4.5$, the total overall anisotropy $A = 3$ and different indicated values of θ^2. Figure 5 presents curves of $f^{\kappa T}$ and f^{κ} versus the energy and pitch angle with the same parameters as those in figure 4. It is found that kappa distribution decays more rapidly than the KT distribution with the energy E_k; especially when θ^2 increases, f^{κ} decays much faster than $f^{\kappa L}$ at higher energies. There is a turning point for each distribution when θ^2 is large;

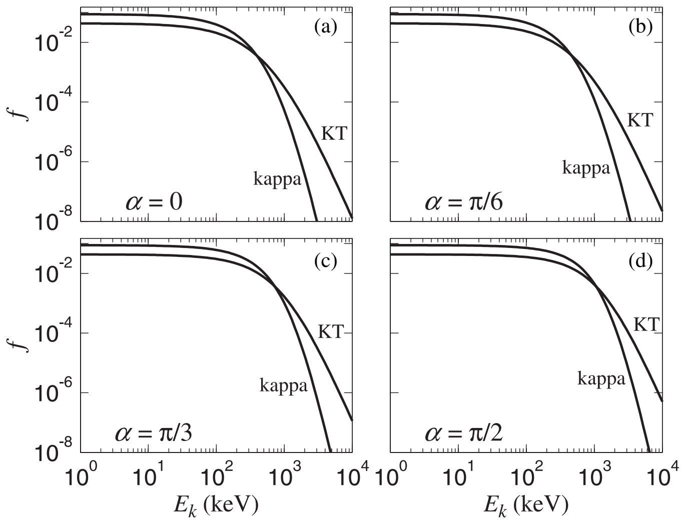

Figure 3. Same as figure 1 except $\theta^2 = 0.3$ (∼150 keV).

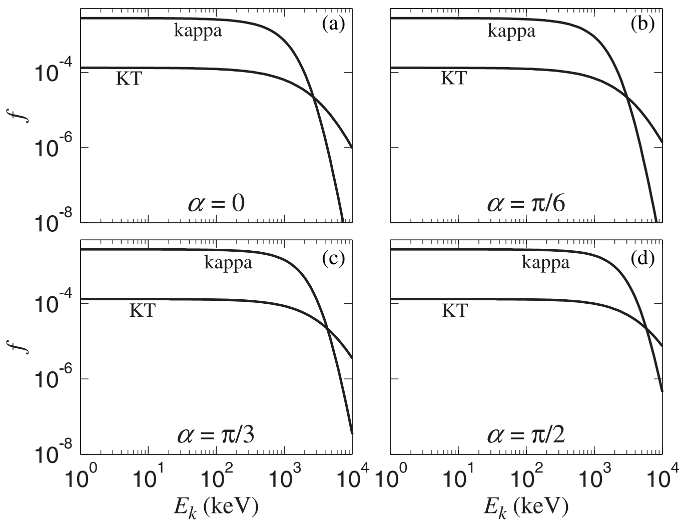

Figure 4. Same as figure 1 except $\theta^2 = 3$ (∼1.5 MeV).

both f^κ and $f^{\kappa T}$ vary slowly before their turning points and then decay rapidly after them. In particular, $f^\kappa/f^{\kappa T} \lesssim 10^{-2}$ for $E_k \gtrsim 2.0$ MeV and $\theta^2 \gtrsim 0.3$. However, no big difference is found between both distributions through energies up to ∼500 keV in the case of $\theta^2 \lesssim 0.03$. The above results suggest that the new KT distribution may obey a more reasonable power-law at relativistic energies since energetic electron spectra do not follow a simple power law but depend on the electron energy range [12, 40].

Furthermore, we shall show the difference between kinetic temperatures $T^{\kappa T}$ and T^κ (defined by $T_\parallel \Rightarrow mc^2 P_\parallel$, and $T_\perp \Rightarrow mc^2 P_\perp/2$) for the KT and kappa distributions and the difference between mean kinetic energies $\overline{E_k^{\kappa T}}$ and $\overline{E_k^\kappa}$ (all obtained by (31)) for the above cases, respectively. In table 1, we present values of $\overline{E_k^{\kappa T}}$ and $\overline{E_k^\kappa}$, $T^{\kappa T}$ and T^κ, respectively, for $\kappa = 6$ and the overall anisotropy $A = 1$. It is found that in the case of $\theta^2 \lesssim 0.03$ the difference

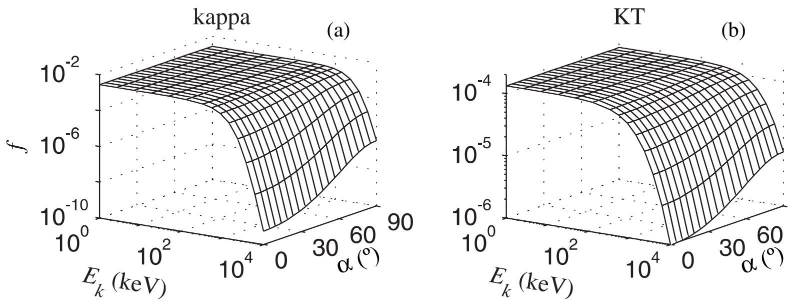

Figure 5. Same as figure 4 except for distribution versus pitch angle and kinetic energy.

Table 1. Scaled mean kinetic energy and temperature values.

θ^2	0.003	0.03	0.3	3	l
$\overline{E_k^{\kappa T}}$	0.01	0.104	1.41	21.1	0
$\overline{E_k^{\kappa}}$	0.01	0.092	0.65	3.15	0
$T_\parallel^{\kappa T}$	0.004	0.047	1.49	195.3	0
$T_\parallel^{\kappa}$	0.004	0.04	0.4	4	0
$\overline{E_k^{\kappa T}}$	0.01	0.11	1.82	29.6	0.5
$\overline{E_k^{\kappa}}$	0.01	0.092	0.67	3.2	0.5
$T_\parallel^{\kappa T}$	0.004	0.049	2.32	388.3	0.5
$T_\parallel^{\kappa}$	0.004	0.04	0.4	4	0.5
$\overline{E_k^{\kappa T}}$	0.01	0.115	2.42	41.7	1
$\overline{E_k^{\kappa}}$	0.01	0.093	0.68	3.3	1
$T_\parallel^{\kappa T}$	0.004	0.05	4.2	871.0	1
$T_\parallel^{\kappa}$	0.004	0.04	0.4	4	1

is very small between each pair of $T^{\kappa T}$ and T^{κ}, $\overline{E_k^{\kappa T}}$ and $\overline{E_k^{\kappa}}$, whereas in the case of $\theta^2 > 0.03$ the difference increases, e.g. $\overline{E_k^{\kappa T}}/\overline{E_k^{\kappa}} > 6.6$ and $T^{\kappa T}/T^{\kappa} > 48$ for $\theta^2 = 3$. This is consistent with the above result.

For the anisotropy behavior, in order to present a direct comparison, we assume the overall anisotropy is the same for the three distribution functions and consider three cases of anisotropy for the new KT (13) and kappa (3) distributions: (a) loss cone, (b) temperature and (c) both loss cone and temperature. The curves for $f^{\kappa L}$, $f^{\kappa T}$ and f^{κ} (left panels) and together with the corresponding normalized f/f_{m} (right panels) for $A^{\kappa L} = A^{\kappa T} = A^{\kappa} = 1$ and $A^{\kappa L} = A^{\kappa T} = A^{\kappa} = 1.5$ are shown in figures 6 and 7, respectively. Here f_{m} is the maximum value at $\alpha = 90^\circ$ corresponding to each of $f^{\kappa L}$, $f^{\kappa T}$ and f^{κ} at their, respectively, indicated values of parameters. It is seen that even for the same overall anisotropy, the KT and kappa distributions either with the temperature anisotropy or with both the loss cone and temperature anisotropy are quite different from the KLC distribution; whereas the three distributions with the same loss cone behaves in a similar way, e.g. curves of the corresponding normalized f/f_{m} merge together in this case (see right panel (f)).

4. Conclusions

In this paper, we have extended the previous study [40] to model energetic particles by a generalized relativistic KT distribution which incorporates either temperature anisotropy

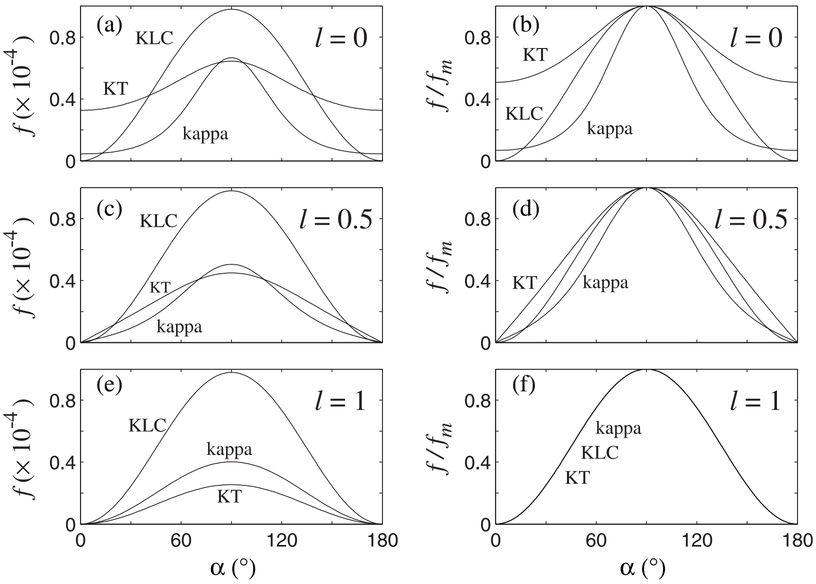

Figure 6. (Left panels) Distribution function curves for $A^{\kappa L} = A^{\kappa T} = A^{\kappa} = 1$, $\kappa = 6$, $p = 8$, $\theta_{\parallel}^2 = 2\theta^2$ and $\theta^2 = 3$. Cases are shown: $l = 0$ (temperature anisotropy), $l = 0.5$ (loss cone and temperature anisotropy) and $l = 1$ (loss cone). (Right panels) Profiles of the corresponding scaled f/f_{m}. Note that curves of $f^{\kappa L}$, $f^{\kappa T}$ and f^{κ} merge together due to the same loss cone (panel (f)).

or both loss cone and temperature anisotropy. Analogous to the recently developed KLC distribution [40], the new KT distribution is found to combine features similar to the well-known KT and loss cone type and also contains features of the relativistic Maxwellian distribution at $\kappa \to \infty$. Numerical calculations are carried out specifically for the energetic magnetospheric electrons with a direct comparison between the new KT distribution, KLC distribution and the regular kappa distribution. The following results are also obtained.

(a) The kappa distribution f^{κ} generally decreases more rapidly than the KT distribution $f^{\kappa T}$; particularly when θ^2 increases, f^{κ} decays much faster than $f^{\kappa L}$ at higher energies. For a large θ^2, a turning point occurs for each distribution after which $f^{\kappa T}$ and f^{κ} drop sharply. Specifically, $f^{\kappa}/f^{\kappa L} \lesssim 10^{-2}$ for $E_k \gtrsim 2.0$ MeV and $\theta^2 \gtrsim 0.3$. However, no big difference occurs between both distributions through energies up to ∼500 keV in the case of $\theta^2 \lesssim 0.03$.

(b) The differences between the mean kinetic energy and temperature for both KT and kappa distributions are found to be small for a small θ^2 but increases as θ^2 increases. For example, $\overline{E_k^{\kappa T}}/\overline{E_k^{\kappa}} > 6.6$ and $T^{\kappa T}/T^{\kappa} > 48$ for $\theta^2 = 3$. This is consistent with the above results.

(c) Even for the same overall anisotropy, the new KT distribution, the previous KLC distribution and the regular kappa distribution differ much from each other. However, the three distributions with the same loss cone behave in a similar way.

In general, since energetic particles present in planetary magnetospheres and other plasmas often exhibit an anisotropy (including loss cone and temperature anisotropy) and a complex power law which depends on the particle energy range [12,40], the new KT distribution which obeys a more physically power law at relativistic energies may be an active tool to model highly energetic particles in the outer radiation belts of the Earth, the Jovian inner magnetosphere and other plasmas (e.g. the laboratory machine) where mirror geometries occur.

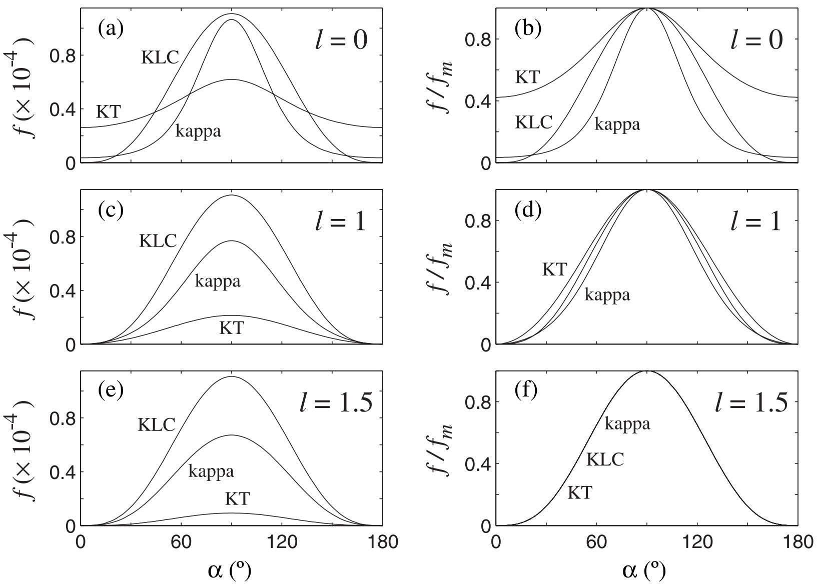

Figure 7. (Left panels) Same as figure 6 but for $A^{\kappa L} = A^{\kappa T} = A^{\kappa} = 1.5$. Cases are shown: $l = 0$ (temperature anisotropy), $l = 1$ (loss cone and temperature anisotropy) and $l = 1.5$ (loss cone). (Right panels) Profiles of the corresponding scaled $f/f_{\rm m}$. Note that curves of $f^{\kappa L}$, $f^{\kappa T}$ and f^{κ} merge together due to the same loss cone (panel (f)).

Acknowledgments

This work is supported by the National Natural Science Foundation of China Grant 40774078, 40774079, 40536029, the Hunan Provincial Natural Science Foundation of China Grant 07JJ3123 and the Visiting Scholar Foundation of State Key Laboratory of Space Weather, CAS.

References

[1] Li X *et al* 2005 *Space Weather* **3** S04001
[2] Zong Q-G and Wilken B 1998 *Geophys. Res. Lett.* **25** 4121
[3] Zong Q-G *et al* 2001 *J. Geophys. Res.* **106** 25541
[4] Leske R A, Mewaldt R A and Stone E C 2001 *J. Geophys. Res.* **106** 30011
[5] Baker D N 2002 *Science* **297** 1486
[6] Li X *et al* 1997 *J. Geophys. Res.* **102** 14123
[7] Li X and Temerin M A 2001 *Space Sci. Rev.* **95** 569
[8] Li X *et al* 2001 *Geophys. Res. Lett.* **28** 1887
[9] Li X 2004 *Space Weather* **2** S03006
[10] Zong Q-G *et al* 2007 *Geophys. Res. Lett.* **34** L12105
[11] Summers D, Thorne R M and Xiao F 1998 *J. Geophys. Res.* **103** 20487
[12] Summers D and Ma C 2000 *J. Geophys. Res.* **105** 2625
[13] Li L, Cao J and Zhou G 2005 *J. Geophys. Res.* **110** A03203
[14] Xiao F *et al* 2006 *J. Geophys. Res.* **111** A11201
[15] Xiao F, Thorne R M and Summers D 2007 *Planet. Space Sci.* **55** 1257
[16] Summers D, Ni B and Meredith N P 2007 *J. Geophys. Res.* **112** A04206
[17] Summers D, Ni B and Meredith N P 2007 *J. Geophys. Res.* **112** A04207
[18] Gary S P and Wang J 1996 *J. Geophys. Res.* **101** 10749
[19] Gary S P, Winske D and Hesse M 2000 *J. Geophys. Res.* **105** 10751
[20] Xiao F *et al* 2006 *J. Geophys. Res.* **111** A08208

[21] Xiao F *et al* 2007 *J. Geophys. Res.* **112** A07219
[22] Galeev A A 1989 *Plasma Waves Instabilities at Comets and in Magnetospheres (AGU Monograph Series)* (Washington DC: American Geophysical Union) p 1–12
[23] Vasyliunas V M 1968 *J. Geophys. Res.* **73** 2839
[24] Christon S P *et al* 1988 *J. Geophys. Res.* **93** 2562
[25] Viñas A F, Mace R L and Benson R F 2005 *J. Geophys. Res.* **110** AO6202
[26] Maksimovic M, Pierrard V and Riley P 1997 *Geophys. Res. Lett.* **24** 1151
[27] Maksimovic M, Pierrard V and Lemaire J F 1997 *Astron. Astrophys.* **324** 725
[28] Saito S *et al* 2000 *Ann. Geophys* **18** 1216
[29] Dasso S, Gratton F T and Farrugia C J 2003 *J. Geophys. Res.* **108** 1149
[30] Freeman J W *et al* 1998 *J. Geophys. Res.* **103** 26251
[31] Burin des Roziers E and Li X 2006 *Space Weather* **4** S06007
[32] Reames D V *et al* 1997 *Astrophys. J.* **483** 515
[33] Summers D and Thorne R M 1991 *Phys. Fluids* B **3** 1835
[34] Summers D, Thorne R M and Matsumoto H 1996 *Phys. Plasmas* **3** 2496
[35] Summers D, Xue S and Thorne R M 1994 *Phys. Plasmas* **1** 2012
[36] Mace R L 1996 *J. Plasma Phys.* **55** 415
[37] Mace R L 2003 *Phys. Plasmas* **10** 2181
[38] Mace R L 2004 *Phys. Plasmas* **11** 507
[39] Mace R L and M A Hellberg 2003 *Phys. Plasmas* **10** 21
[40] Xiao F 2006 *Plasma Phys. Control. Fusion* **48** 203
[41] Xiao F *et al* 2008 *Plasma Phys. Control. Fusion* **50** 062001
[42] Xiao F *et al* 2008 *J. Geophys. Res.* **113** A05203
[43] Dory R A Guest G E and Harris G E 1965 *Phys. Rev. Lett.* **14** 131
[44] Xiao F, Thorne R M and Summers D 1998 *Phys. Plasmas* **5** 2489
[45] Mace R L 1996 *Phys. Scr.* T **63** 207
[46] Mace R L 1998 *J. Geophys. Res.* **103** 14643
[47] Gladd N T 1983 *Phys. Fluids* **26** 974
[48] Schlickeiser R, Fichtner H and Kneller M 1997 *J. Geophys. Res.* **102** 4725