Space Plasma Physics Fall 2024

Problem Set 1

Due date: Nov. 8, 2024

1. Distribution function and averages. The average of a quantity G(**v**) over a distribution function f(**v**) is defined as

$$< \mathrm{G} >= \frac{\int G(\boldsymbol{v})\, f(\boldsymbol{v}) d^3\boldsymbol{v}}{\int f(\boldsymbol{v})\, d^3\boldsymbol{v}}$$

The Maxwellian distribution (in 3D) is

$$\mathrm{f}(\boldsymbol{v}) = \mathrm{n}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2T}\right)$$

Note that the isotropy of this distribution means that all Cartesian coordinates are equivalent, there is no preferred direction.

(a) Prove that the form of f is correctly normalized, i.e. that $\int f d^3\boldsymbol{v} = n$

Evaluate the averages of the following:

(b) a specific Cartesian coordinates: $< v_x >$

(c) the square velocity: $< v^2 >$, and the average particle energy $< \frac{1}{2}\mathrm{m}v^2 >$

(d) the average speed $< |\boldsymbol{v}| >$

2. Show for a steady-state flow ($\frac{\partial}{\partial t} = 0$) that the momentum equation with a gravitational force, $F_g = \rho_m \boldsymbol{g}$, an isotropic pressure, and no electromagnetic forces ($\boldsymbol{E}$ = 0 and $\boldsymbol{B}$ = 0)

$$\rho_m \left[\frac{\partial V}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right] = -\nabla P + \rho_m \boldsymbol{g}$$

can be integrated once along a streamline to obtain Bernoulli's equation:

$$\frac{1}{2}\rho_m V^2 + P + \rho_m g z = constant$$

Assume that the gravitational force term can be derived from a potential, $\rho_m \boldsymbol{g} = -\boldsymbol{\nabla} \emptyset_{\boldsymbol{g}}$ where $\emptyset_g = \rho_m g z$ and g is constant.

3. The energy equation for the ideal MHD plasma. In the limit of ideal MHD, the MHD equations conserve energy exactly. To derive an expression for the energy in this limiting case, let us start with the momentum equation. Multiplying the momentum equation by **V,** and eliminating **J** via Ampere's law, show that:

(a) $\rho_m \boldsymbol{V} \cdot \left[\frac{\partial V}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right] = -(\boldsymbol{V} \cdot \nabla)P + \frac{V}{\mu_0} \cdot [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]$

Using the mass continuity equation, show that the left-hand side of the above equation can be rewritten in the following form:

(b) $\rho_m \boldsymbol{V} \cdot \left[\frac{\partial V}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right] = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m V^2\right) + \nabla \cdot \left(\frac{V^2}{2}\rho_m \boldsymbol{V}\right)$

Use the adiabatic equation of state, i.e. $\frac{d}{dt}\left(\frac{P}{\rho_m{}^\gamma}\right) = 0$, show that the first term on the right-hand of equation (a) can be solved in the following the form:

(c) $(\mathbf{V} \cdot \nabla)\mathrm{P} = \frac{1}{\gamma-1}\frac{\partial P}{\partial t} + \frac{\gamma}{\gamma-1}\nabla \cdot (P\boldsymbol{V})$

The second term on the right-hand of equation (a) can we rewritten as $\mathbf{V} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} = -(\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \mathbf{B})$. Noting that $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ for an ideal MHD fluid, show the second term can be rewritten as:

(d) $\mathbf{V} \cdot (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B})$

Finally, substituting equations (b) (c)(d) into equation (a), we obtain the energy conservation law for an ideal MHD fluid. Please write down the correct energy equation.