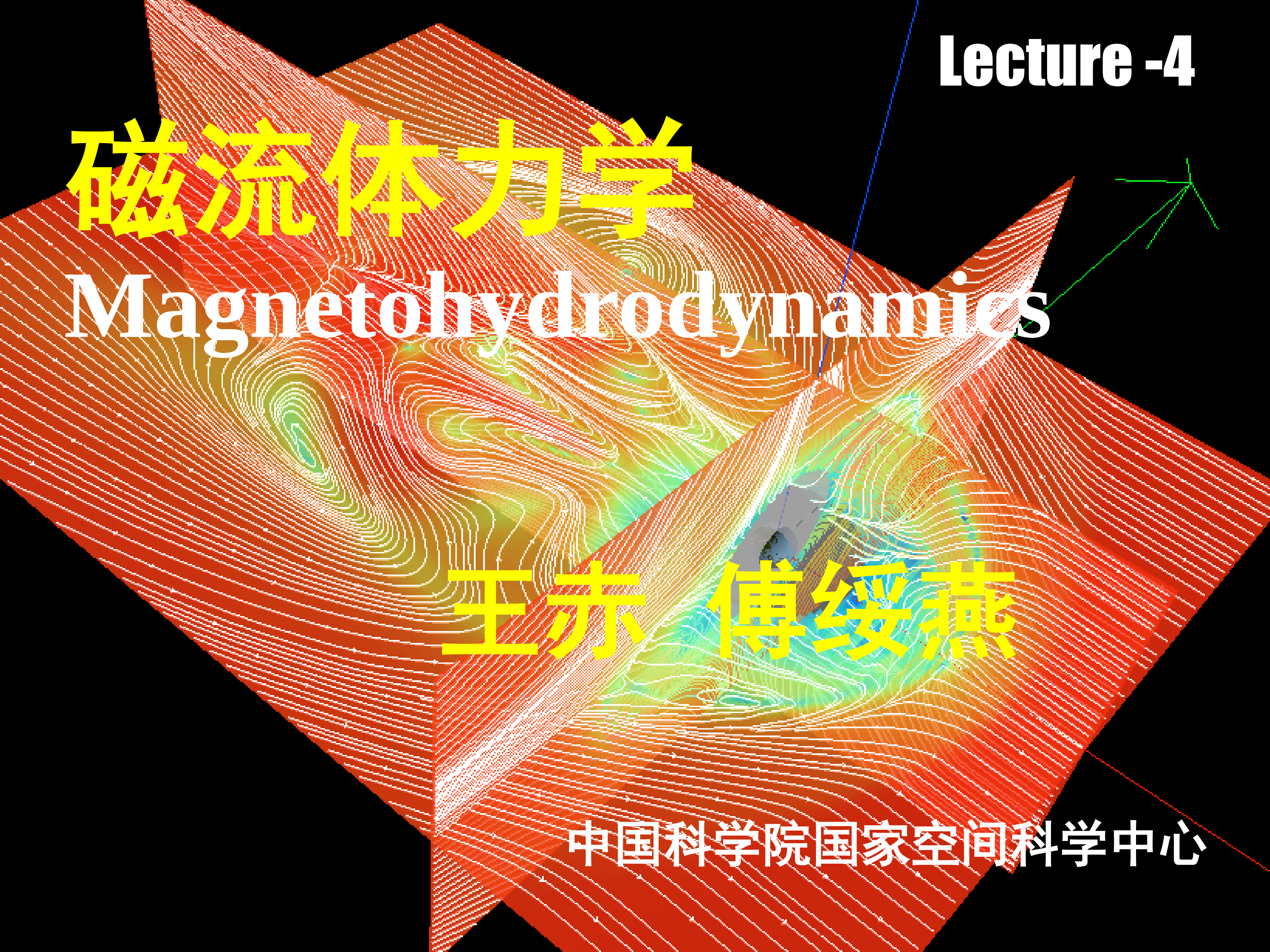
Lecture -4

磁流体力学

Magnetohydrodynamics



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MHD Waves

- Alfven mode
- Fast and slow magnetosonic modes

Representation of Waves

Any periodic motion of a fluid can be decomposed by Fourier analysis into a superposition of sinusoidal oscillations with different frequencies ω and wavelengths λ

When the oscillation amplitude is small, the waveform is general sinusoidal; and there is only one component. This is the situation we shall consider.

Any sinusoidal oscillating quantity, say , the density n - , can be represented as follows: $n = n_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \boldsymbol{\omega} t)]$

The time derivative $\partial/\partial t$ can therefore be replaced by $-i\omega$, and the gradient ∇ by $i\boldsymbol{k}$

Sound Wave

For a neutral fluid like air, in absence of viscosity, the Navier-Stokes equation is

$$\rho_m\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p$$

From the equation of state

$$p\rho_m^{-\gamma}=const \qquad \nabla p=\frac{\gamma p}{\rho_m}\nabla\rho_{\mathrm{m}}$$

then

$$\rho_m\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\frac{\gamma p}{\rho_m}\nabla\rho_{\mathrm{m}}$$

Continuity equation yields

$$\frac{\partial\rho_m}{\partial t}+\nabla\cdot(\rho_m\mathbf{u})=0$$

Linearization of the momentum and continuity equations for stationary equilibrium

$\rho_m = \rho_0 + \rho_1 \qquad \rho_0 >> \rho_1$

$p = p_0 + p_1 \qquad p_0 >> p_1$

$\vec{\mathrm{u}} = \vec{\mathrm{u}}_0 + \vec{\mathrm{v}}_1 = \vec{\mathrm{v}}_1$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \left(\rho_m \mathbf{u} \right) = 0$$

$$\frac{\partial (\rho_0 + \rho_1)}{\partial t} + \nabla \cdot \left((\rho_0 + \rho_1) \vec{\mathbf{v}}_1 \right) = 0$$

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \left(\rho_0 \vec{\mathbf{v}}_1 + \rho_1 \vec{\mathbf{v}}_1 \right) = 0$$

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \left(\rho_0 \vec{\mathbf{v}}_1 \right) = 0$$

$$-i\omega \rho_1 + \rho_0 i \vec{\mathrm{k}} \cdot \vec{\mathrm{v}}_1 = 0$$

$$\rho_m\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\frac{\gamma p}{\rho_m}\nabla\rho_{\mathrm{m}}$$

$$(\rho_0+\rho_1)\left[\frac{\partial \vec{\mathbf{v}}_1}{\partial t}+(\vec{\mathbf{v}}_1\cdot\nabla)\vec{\mathbf{v}}_1\right]=-\nabla(p_0+p_1)$$

LHS: $$\left[\rho_0\frac{\partial \vec{\mathbf{v}}_1}{\partial t}+\rho_0(\vec{\mathbf{v}}_1\cdot\nabla)\vec{\mathbf{v}}_1\right]+\left[\rho_1\frac{\partial \vec{\mathbf{v}}_1}{\partial t}+\rho_1(\vec{\mathbf{v}}_1\cdot\nabla)\vec{\mathbf{v}}_1\right]=\rho_0\frac{\partial \vec{\mathbf{v}}_1}{\partial t}$$

RHS: $$\nabla(p_0+p_1)=-\frac{\gamma(p_0+p_1)}{(\rho_0+\rho_1)}\nabla(\rho_0+\rho_1)=-\frac{\gamma p_0}{\rho_0}\nabla(\rho_0+\rho_1)$$

$$-i\omega\rho_0\vec{\mathrm{v}}_1=-\frac{\gamma p_0}{\rho_0}i\vec{\mathrm{k}}\rho_1$$

$$-i\omega\rho_1+\rho_0 i\vec{\mathrm{k}}\cdot\vec{\mathrm{v}}_1=0$$

$$-i\omega\rho_0\vec{\mathrm{v}}_1=-\frac{\gamma p_0}{\rho_0}i\vec{k}\,\frac{\rho_0 i\vec{k}\cdot\vec{\mathrm{v}}_1}{i\omega}$$

For a plane wave with **k** = k**x**, and **v** = v**x**, we find

$$\frac{\omega}{k} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k_B T}{m}\right)^{1/2} = c_s$$

where m is the neutral atom mass and Cs is the sound speed.

- For a neutral gas the sound waves are pressure waves propagating from one layer of particles to another one. Wave vector pointing normal to the pressure front.
- The propagation of sound waves requires collisions among the neutrals
- Magnetic field does not affect motion parallel to the field. Therefore sound waves can propagate with c_s at k//B

Alfven Waves

Studying disturbances propagating on the solar surface:
----Puzzle: not consistent with c_s
----Conclusion: unknown wave mode exits in plasma.

- MHD is a fluid theory and there are similar wave modes as in ordinary fluid theory (hydrodynamics).
- In hydrodynamics the restoring forces for perturbations are the pressure gradient and gravity.
- In MHD the pressure force leads to acoustic fluctuations, whereas Ampère's force (**J**x**B**) leads to an entirely new class of wave modes, called Alfvén (or MHD) waves.

Existence of Electromagnetic-Hydrodynamic Waves

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IF a conducting liquid is placed in a constant magnetic field, every motion of the liquid gives rise to an E.M.F. which produces electric currents. Owing to the magnetic field, these currents give mechanical forces which change the state of motion of the liquid.

Thus a kind of combined electromagnetic-hydrodynamic wave is produced which, so far as I know, has as yet attracted no attention.

The phenomenon may be described by the electrodynamic equations

$$\text{rot } H = 4\frac{\pi}{c} i$$

$$\text{rot } E = -\frac{1}{c}\frac{dB}{dt}$$

$$B = \mu H$$

$$i = \sigma(E + \frac{v}{c} \times B);$$

together with the hydrodynamic equation

$$\partial \frac{dv}{dt} = \frac{1}{c}(i \times B) - \text{grad } p,$$

where σ is the electric conductivity, μ the permeability, δ the mass density of the liquid, *i* the electric current, *v* the velocity of the liquid, and *p* the pressure.

Consider the simple case when σ = }, μ = 1 and the imposed constant magnetic field H_0 is homogeneous and parallel to the *z*-axis. In order to study a plane wave we assume that all variables depend upon the time *t* and *z* only. If the velocity *v* is parallel to the *x*-axis, the current *i* is parallel to the *y*-axis and produces a variable magnetic field *H* 2 in the *x*-direction. By elementary calculation we obtain

$$\frac{d^2 H'}{dz^2} = \frac{4\pi \partial}{H_0^2}\frac{d^2 H'}{dt^2},$$

which means a wave in the direction of the *z*-axis with the velocity

$$V = \frac{H_0}{\sqrt{4\pi \partial}}$$

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and as solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have H_0 = 15 gauss and δ = 0·005 gm. cm.$^{-3}$, the velocity of the waves amounts to

$$V \sim 60 \text{ cm. sec.}^{-1}.$$

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of H_0 and δ refer to a distance of about 10_{10} cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance proceeding as an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in *Arkiv för matematik, astronomi och fysik*.

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Aug. 24.

Ampère's force

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B}$$

$$\mathbf{J}\times\mathbf{B} = \frac{1}{\mu_0}(\mathbf{B}\bullet\nabla)\mathbf{B} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

- Suppose that the magnetic field is approximately uniform, and directed along the z-axis. The equation of motion reduces to

$$\rho \frac{d\boldsymbol{V}}{dt} = -\nabla\cdot\boldsymbol{P},$$

$$\boldsymbol{P} = \begin{pmatrix} p + B^2/2\mu_0 & 0 & \\ 0 & p + B^2/2\mu_0 & 0 \\ 0 & 0 & p - B^2/2\mu_0 \end{pmatrix}.$$

- The magnetic field increases the plasma pressure in perpendicular directions to the magnetic field, and decreases the plasma pressure, by the same amount, in the parallel direction.
- Thus, the magnetic field gives rise to a magnetic pressure, acting perpendicular to field-lines, and a magnetic tension acting along field-lines.

➢the parallel direction (tension force)

- Simple stretching of field lines
  $\Rightarrow$ stress reduces to a tension B^2/μ_0 along the magnetic field.
- Tension acts to restore a magnetic field line being stretched

$\sim$

tension in a guitar string being plucked.

$\Rightarrow$ transverse, parallel-propagating magnetic disturbance with the speed

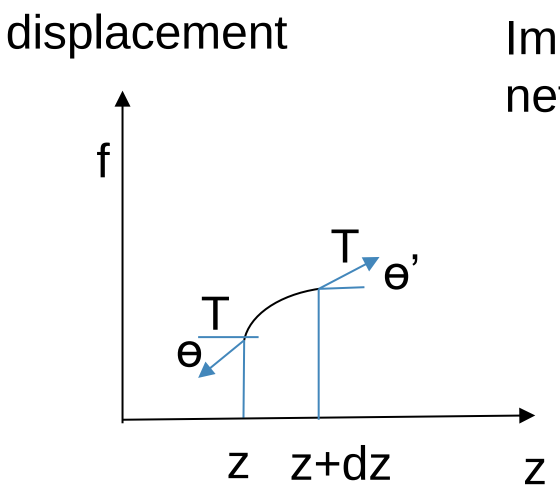
$$v_A = \left(\frac{\text{tension}}{\text{mass density}}\right)^{1/2} = \frac{B}{\sqrt{\mu_0 \rho_m}}$$

(*See Next Page)

analogous to a mechanical disturbance in a string.
v_A is called the *Alfvén speed*.

➢Alfven wave propagate in the parallel direction

Speed of Propagation



Imagine a very long string under Tension T. The net transverse force on the segment z and z+dz, is

$$dF = T\sin\theta' - T\sin\theta$$

Provided that the distortion of the string is not too great, these angles are small, and we can replace the sine by the tangent:

$$df \cong T(\tan\theta' - \tan\theta) = T\left(\frac{\partial f}{\partial z}\bigg|_{z+dz} - \frac{\partial f}{\partial z}\bigg|_{z}\right) = T\frac{\partial^2 f}{\partial z^2}dz$$

If the mass per unit length is μ, Newton's second law says:

$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$ and therefore $\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}$

Evidently, small disturbance on the string satisfy

$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$, where $v = \sqrt{\frac{T}{\mu}}$, representing the speed of propagation.

the perpendicular direction (pressure force)

For motion perpendicular to the magnetic field, in addition to the kinetic fluid pressure *p*, there is also the magnetic pressure.

- Relevant pressure gradient is that of the total pressure

$$\nabla(p+\frac{B^2}{2\mu_0})=v_{ms}^2\nabla\rho_m$$

- Magnetic Flux freezing:

Perpendicular B-compression corresponding to density compression, the magnetic flux B dS across an element of surface ds (whose normal is orientated along the magnetic field) and the mass ρ_m dS of a unit length of column having dS as base are both conserved during the mass motion.

$$B/\rho_m = const \qquad p/\rho_m^{\gamma} = const$$

- Gradient now becomes

$$\nabla\left(p+\frac{B^2}{2\mu_0}\right)= {v_{ms}}^2\nabla\rho_m = \frac{\gamma p}{\rho_m}\nabla\rho_m + \frac{B^2}{\mu_0\rho_m}\nabla\rho_m$$

- We have

$${v_{ms}}^2 = {C_s}^2 + {V_a}^2$$

Magnetosonic waves

Propagate in the perpendicular direction

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.
-> Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.
-> Then all variables can be expressed by one variable

Usually, in space plasma the background magnetic field is sufficiently strong, so that one can assume the fluctuation obeys:

$$|\delta \mathbf{B}| \ll B_0$$

In the uniform plasma with straight field lines, the field provides the only **symmetry axis** which may be chosen as z-axis of the coordinate system such that: $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{||}$.

Eq. of ideal MHD in an adiabatic case

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0$$

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = \mathbf{j} \times \mathbf{B} - \nabla p$$

$$\nabla p = \frac{\gamma p}{\rho_m} \nabla \rho_m$$

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -\frac{\gamma p}{\rho_m} \nabla \rho_m + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})$$

$$\nabla \bullet \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

- Assume that the equilibrium state is static ($\vec{V}_0 = 0$) and homogeneous (ρ_{m0}, p_0 and $\vec{B}_0$ constants), and write the variables as

$$
\begin{aligned}
\rho_m &= \rho_{m0} + \rho_{m1}(\vec{r}, t) \\
\vec{V} &= \vec{V}_1(\vec{r}, t) \\
p &= p_0 + p_1(\vec{r}, t) \\
\vec{B} &= \vec{B}_0 + \vec{B}_1(\vec{r}, t),
\end{aligned}
$$

where $X_1 \ll X_0$ for $X = \rho_m,\ p,\ |\vec{B}|$.

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{\mathbf{V}}) = 0$$

$$\frac{\partial \rho_{m1}}{\partial t} + \rho_{m0} \nabla \cdot \vec{\mathbf{V}}_1 = 0$$

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\frac{\gamma p}{\rho_m} \nabla \rho_m + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0}$$

$$\nabla p_1 = v_s^2 \nabla \rho_{m1}$$

$$v_s^2 = \frac{\gamma p_0}{\rho_{m0}}$$

$$\rho_{m0} \frac{\partial \vec{\mathbf{V}}_1}{\partial t} + v_s^2 \nabla \rho_{m1} + \frac{\vec{\mathbf{B}}_0 \times (\nabla \times \vec{\mathbf{B}}_1)}{\mu_0} = 0$$

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = \nabla \times (\vec{\mathbf{V}} \times \vec{\mathbf{B}})$$

$$\frac{\partial \vec{\mathbf{B}}_1}{\partial t} - \nabla \times (\vec{\mathbf{V}}_1 \times \vec{\mathbf{B}}_0) = 0$$

MHD wave equation

$$\rho_{m0}\frac{\partial \vec{\mathbf{V}}_1}{\partial t}+v_s^2\nabla\rho_{m1}+\frac{\vec{\mathbf{B}}_0\times(\nabla\times\vec{\mathbf{B}}_1)}{\mu_0}=0$$

$$\rho_{m0}\frac{\partial^2 \vec{\mathbf{V}}_1}{\partial t^2}+v_s^2\nabla\frac{\partial}{\partial t}\rho_{m1}+\frac{\vec{\mathbf{B}}_0\times(\nabla\times\frac{\partial}{\partial t}\vec{\mathbf{B}}_1)}{\mu_0}=0$$

$$\frac{\partial\rho_{m1}}{\partial t}+\rho_{m0}\nabla\cdot\vec{\mathbf{V}}_1=0$$

$$\frac{\partial\vec{\mathbf{B}}_1}{\partial t}-\nabla\times(\vec{\mathbf{V}}_1\times\vec{\mathbf{B}}_0)=0$$

$$\vec{\mathbf{v}}_A=\frac{\vec{\mathbf{B}}_0}{\sqrt{\mu_0\rho_{m0}}}$$

$$\frac{\partial^2\vec{\mathbf{V}}_1}{\partial t^2}-v_s^2\nabla(\nabla\cdot\vec{\mathbf{V}}_1)+\vec{\mathbf{v}}_A\times\{\nabla\times[\nabla\times(\vec{\mathbf{V}}_1\times\vec{\mathbf{v}}_A)]\}=0$$

Let us seek for plane wave solution in form:

$$\vec{\mathbf{V}}_1(\vec{\mathbf{r}},t) = \vec{\mathbf{V}}_1 \exp\{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)\}$$

$$\frac{\partial^2 \vec{\mathbf{V}}_1}{\partial t^2} - v_s^2 \nabla(\nabla\cdot\vec{\mathbf{V}}_1) + \vec{\mathbf{v}}_A \times \{\nabla\times[\nabla\times(\vec{\mathbf{V}}_1\times\vec{\mathbf{v}}_A)]\} = 0$$

$$\partial/\partial t \rightarrow -i\omega,\ \nabla \rightarrow i\boldsymbol{k}$$

$$-\omega^2\vec{\mathbf{V}}_1 + v_s^2\vec{\mathbf{k}}(\vec{\mathbf{k}}\cdot\vec{\mathbf{V}}_1) - \vec{\mathbf{v}}_A\times\{\vec{\mathbf{k}}\times[\vec{\mathbf{k}}\times(\vec{\mathbf{V}}_1\times\vec{\mathbf{v}}_A)]\} = 0$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$$

$$-\omega^2\vec{\mathbf{V}}_1 + (v_s^2 + v_A^2)(\vec{\mathbf{k}}\cdot\vec{\mathbf{V}}_1)\vec{\mathbf{k}} +$$
$$(\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)[(\vec{\mathbf{k}}\cdot\vec{\mathbf{v}}_A)\vec{\mathbf{V}}_1 - (\vec{\mathbf{V}}_1\cdot\vec{\mathbf{v}}_A)\vec{\mathbf{k}} - (\vec{\mathbf{k}}\cdot\vec{\mathbf{V}}_1)\vec{\mathbf{v}}_A] = 0$$

I. Propagation perpendicular to the B_0 field

$$\mathbf{k} \perp \mathbf{B}_0$$

$$(\vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_A) = 0$$

$$-\omega^2 \vec{\mathbf{V}}_1 + (v_s^2 + v_A^2)(\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{k}} + (\vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_A)[(\vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_A)\vec{\mathbf{V}}_1 - (\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{v}}_A)\vec{\mathbf{k}} - (\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{v}}_A] = 0$$

$$\Longrightarrow \quad -\omega^2 \vec{\mathbf{V}}_1 + (v_s^2 + v_A^2)(\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{k}} = 0$$

$$\Longrightarrow \quad \vec{\mathbf{V}}_1 = \frac{v_{ms}^2}{\omega^2}(\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{k}}$$

$$\vec{\mathbf{k}} // \vec{\mathbf{V}}_1$$

$$\Longrightarrow \quad \left(\frac{\omega}{k}\right)^2 = v_{ms}^2 = v_s^2 + v_A^2$$

$$\frac{\partial \rho_{m1}}{\partial t} + \rho_{m0} \nabla \cdot \vec{\mathbf{V}}_1 = 0$$

$$-i\omega\rho_{m1} + \rho_{m0} i\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1 = 0$$

$$\rho_{m1} = \rho_{m0} \frac{V_1}{\omega / k}$$

$$p_1 = v_s^2 \rho_{m1} = \frac{\gamma V_1}{\omega / k} p_0$$

1. Magnetic perturbation follows the induction equation as

$$\frac{\partial \vec{\mathbf{B}}_1}{\partial t} - \nabla \times (\vec{\mathbf{V}}_1 \times \vec{\mathbf{B}}_0) = 0 \longrightarrow -i\omega \mathbf{B}_1 - i\mathbf{k} \times (\mathbf{V}_1 \times \mathbf{B}_0) = 0$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_0 = 0 \longrightarrow \boldsymbol{B}_1 = \frac{-\boldsymbol{k} \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0)}{\omega} = \frac{V_1}{(\omega/k)} \boldsymbol{B}_0$$

2. Wave Electric field from the Ohm's law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} = \mathbf{B}_0 \times \mathbf{V}_1$$

3. Linearity means : $|\mathbf{B}_1| << |\mathbf{B}_0| \longrightarrow V_1 << |\omega / k| = v_{ms}$

4. This wave resembles ordinary electromagnetic waves in that **k**, $\mathbf{B}_1$ and $\mathbf{E}_1$ are all perpendicular to each other

5. This wave is called a magnetosonic or a magneto-acoustic wave.

$\mathbf{V}_1 // \mathbf{k}$ Compressional or a fast Alfven wave

II. Propagation parallel to the B_0 field

$$\vec{\mathbf{k}} // \vec{\mathbf{B}}_0 // \vec{\mathrm{v}}_A$$

$$-\omega^2 \vec{\mathbf{V}}_1 + (v_s^2 + v_A^2)(\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{k}} + (\vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_A)[(\vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_A)\vec{\mathbf{V}}_1 - (\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{v}}_A)\vec{\mathbf{k}} - (\vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1)\vec{\mathbf{v}}_A] = 0$$

$$(k^2 v_A^2 - \omega^2)\vec{\mathbf{V}}_1 + (\frac{v_s^2}{v_A^2} - 1)k^2 (\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{v}}_A)\vec{\mathbf{v}}_A] = 0$$

It has two solutions:

$$(k^2 v_s^2 - \omega^2)\vec{\mathbf{V}}_{1//} = 0$$

$$(k^2 v_A^2 - \omega^2)\vec{\mathbf{V}}_{1\perp} = 0$$

$\mathbf{k} // \mathbf{B}_0$ Case 1: $\vec{\mathbf{V}}_1 // \vec{\mathbf{v}}_A$

$$(k^2 v_s^2 - \omega^2)\vec{\mathbf{V}}_{1//} = 0$$

Dispersion relation:

$$\omega^2 / k^2 = v_s^2$$

Sound wave

1. Density and pressure perturbation

$$\rho_{m1} = \rho_{m0} \frac{V_1}{\omega / k}$$

$$p_1 = \frac{\gamma V_1}{\omega / k} p_0$$

$$\rho_{m1} = \rho_{m0} \frac{V_1}{v_s}$$

$$p_1 = \frac{\gamma V_1}{v_s} p_0$$

2. No electric and magnetic perturbation

$$\boldsymbol{B}_1 = \frac{-\boldsymbol{k} \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0)}{\omega} = 0$$

$$\mathbf{E} = \mathbf{B}_0 \times \mathbf{V}_1 = 0$$

$\mathbf{k}//\mathbf{B}_0$ Case 2: $\vec{\mathbf{V}}_1 \perp \vec{\mathbf{v}}_A$

$$(k^2 v_A^2 - \omega^2)\vec{\mathbf{V}}_{1\perp} = 0$$

Dispersion relation: $\omega^2 / k^2 = v_A^2$ Alfven wave

1. No density perturbation

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \vec{\mathbf{V}}) = 0$$

$$-\omega \rho_{m1} + \rho_{m0} \vec{\mathbf{k}} \cdot \vec{\mathbf{V}}_1 = 0 \qquad \rho_{m1} = 0 \qquad p_1 = 0$$

2. electric and magnetic perturbation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \qquad \boldsymbol{B}_1 = \frac{-\boldsymbol{k} \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0)}{\omega} = \frac{-B_0}{(\omega/k)} \boldsymbol{V}_1 \qquad \mathbf{E} = -\mathbf{V}_1 \times \mathbf{B}_0$$

3. Shear Alfven wave $\mathbf{E} \perp \mathbf{k}$

4. Velocity disturbance is perpendicular to the back ground magnetic field and wave vector. $\mathbf{V}_1 \perp \mathbf{k}$ $\mathbf{V}_1 \perp \mathbf{B}_0$

Alfven wave with ***k*** parallel to $\boldsymbol{B}_0$.

Magnetic field is not compressed, only undulating ==> no density disturbance (frozen-in condition).

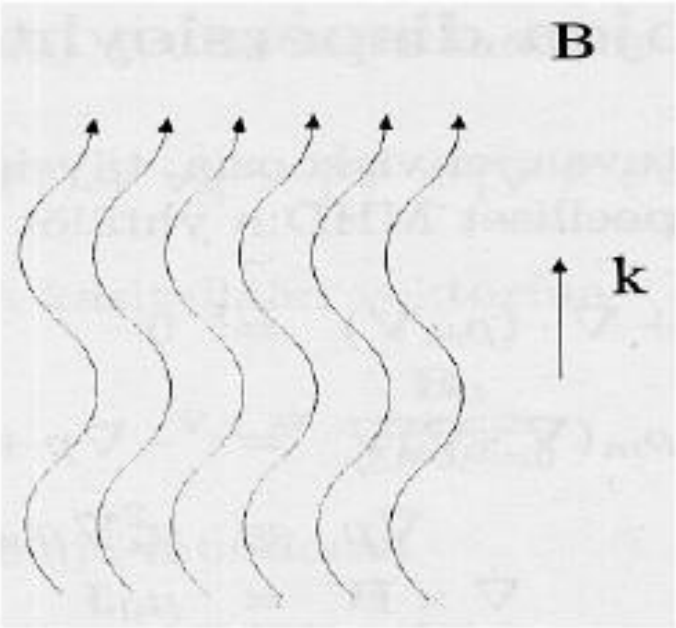
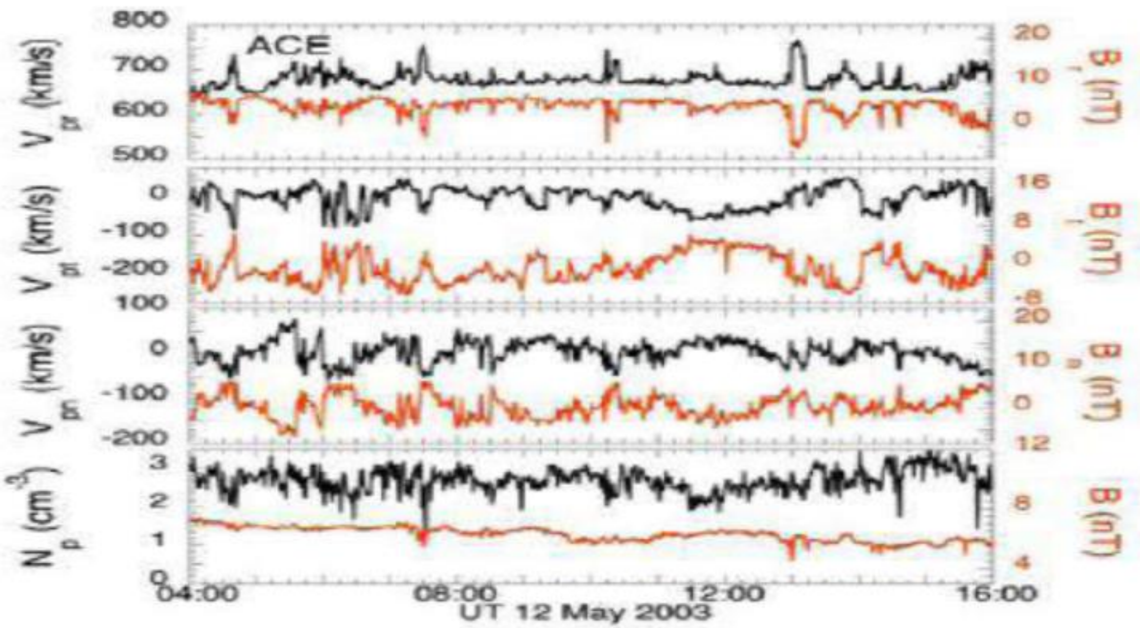


Figure from Koskinen (2001)

Alfven Waves in the solar wind

For Alfven waves, $B_1 = \frac{-B_0}{(\omega/k)} V_1 \rightarrow \frac{V_1}{B_1} = -\frac{V_a}{B_0}$

The relation given by the above equations is very important to identify Alfven waves in the solar wind. The amplitude of velocity and magnetic field is correlated and the difference between their phases is 180 degrees.



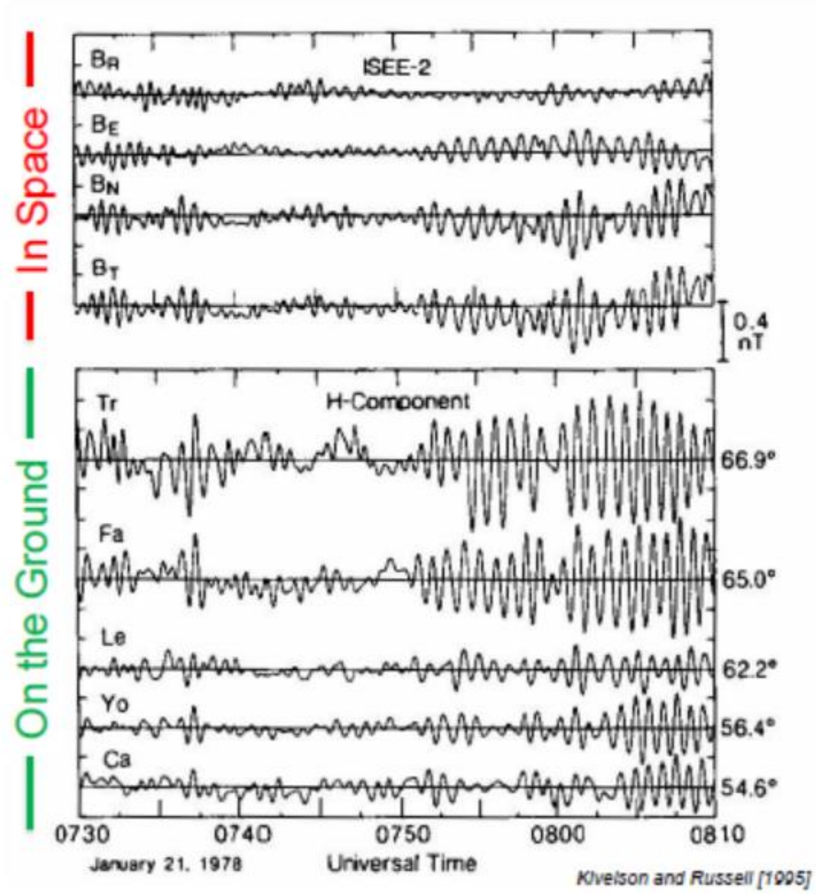
(Gosling et al. 2009, ApJ)

Geomagnetic Pulsations

First published account in Stewart[1961]
– Quasi-sinusoidal magnetic field oscillations
– Field changes on the order of 100nT
– Time scale of a few minutes

What we could now call a resonant MHD standing wave
– aka a ultra-low frequency (ULF) wave
– aka a field-line resonance (FLR)

First theoretical treatment in Dungey (1954)
Experimental verification of Dungey's theory came from ground and satellite measurements in the 1960s and 1970s



Resonant MHD Standing Waves in the Magnetosphere

- Make serval simplifying assumptions

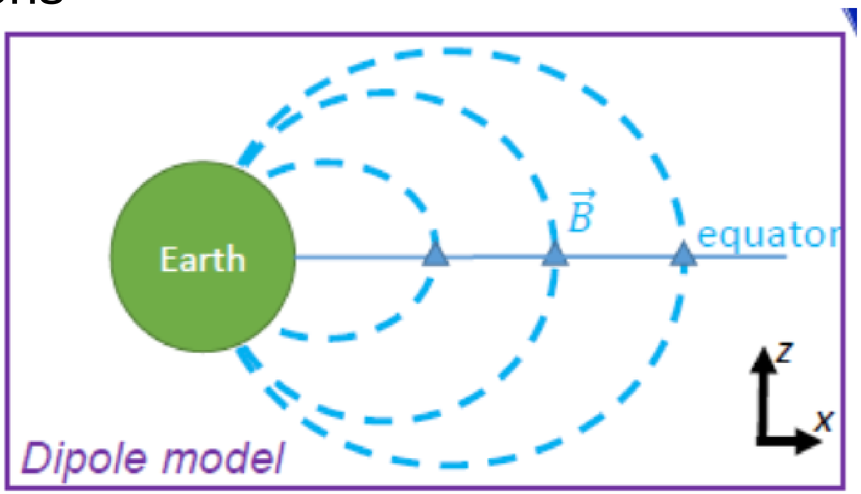
– Cold plasma

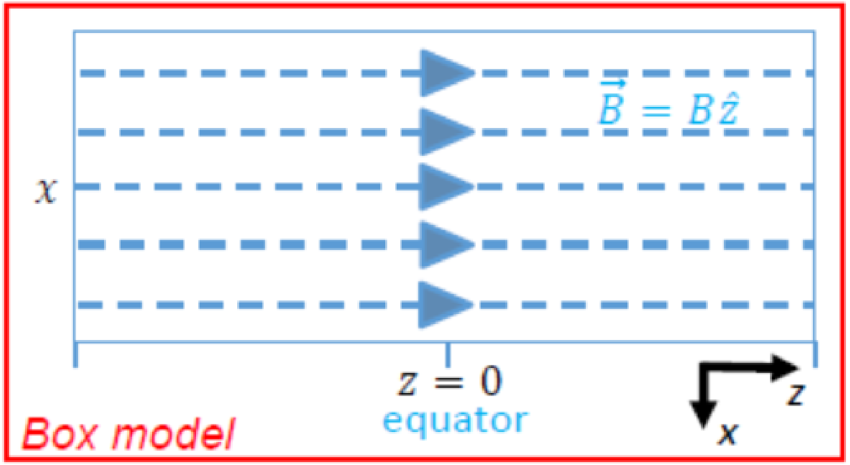
– Uniform plasma

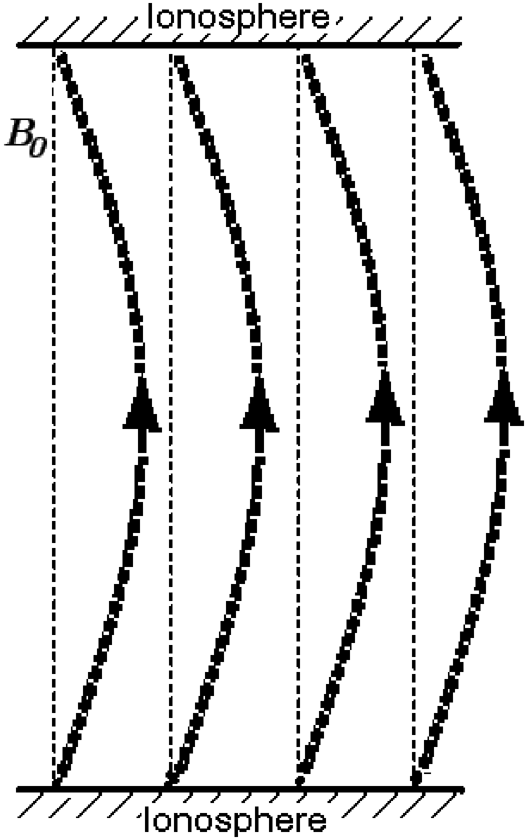
– Uniform magnetic field

- Linearize

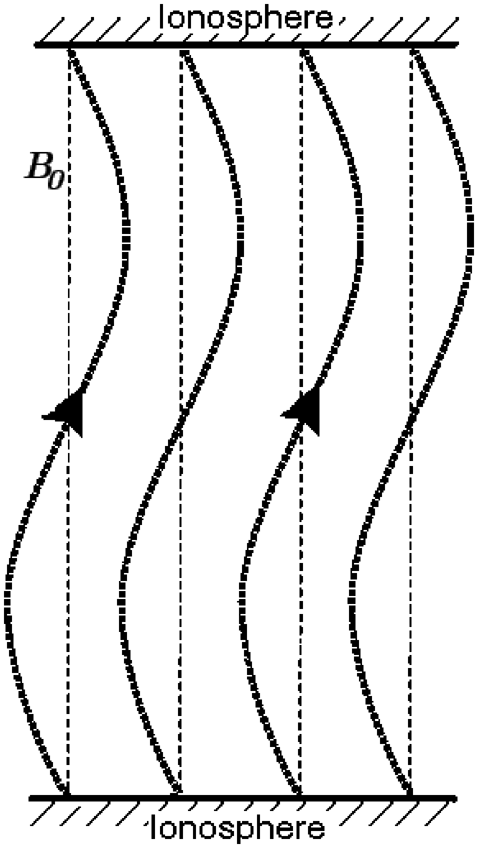
$$\frac{\partial^2 \vec{E}}{\partial t^2} + V_A^2\, \nabla \times \left(\nabla \times \vec{E}\right) = 0$$







Alfven Standing waves
（fundamental）

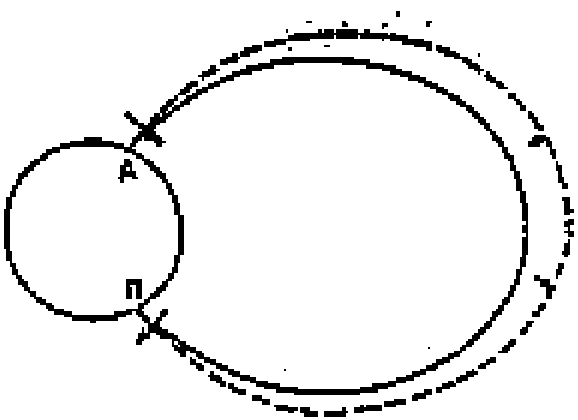


Alfven standing waves
（second harmonic）

Only Certain resonant frequencies can be established. These frequencies are controlled by the length of field lines between the ionospheres, the strength of the magnetic field, and the plasma density.

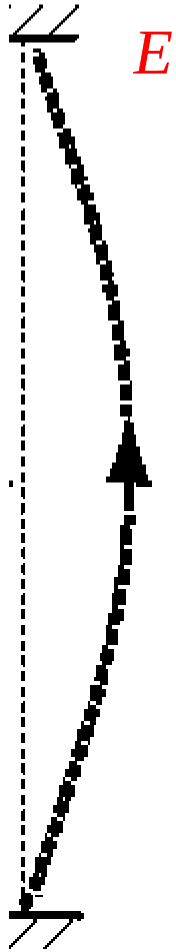
$$f = \frac{nV_A}{2l} \equiv \frac{nB_0}{2l\sqrt{\mu\rho}}$$

Fundamental

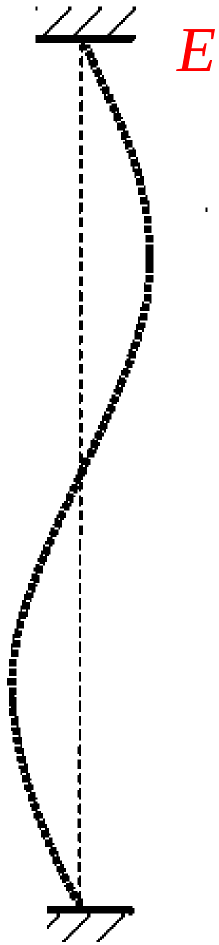


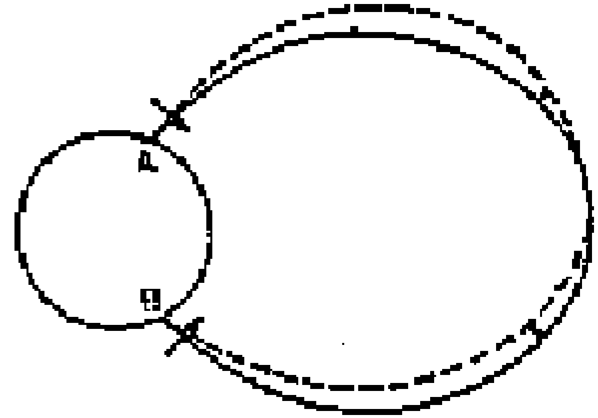
poloidal mode

toroidal mode

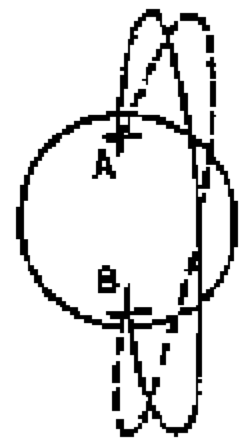


Second Harmonic





poloidal mode



toroidal mode

III. Oblique Propagation

- Choose the coordinate system so that $\vec{B}_0 = B_0\vec{e}_z$, $\vec{k} = k_x\vec{e}_x + k_z\vec{e}_z$.
- Denote the angle $\sphericalangle(\vec{B}_0, \vec{k}) \equiv \theta$. Thus,

$$
\begin{aligned}
\vec{k} &= k(\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_z) \\
\vec{v}_A &= v_A\vec{e}_z \\
\vec{V}_1 &= V_{1x}\vec{e}_x + V_{1y}\vec{e}_y + V_{1z}\vec{e}_z \\
\vec{k}\cdot\vec{v}_A &= kv_A\cos\theta \\
\vec{k}\cdot\vec{V}_1 &= k(V_{1x}\sin\theta + V_{1z}\cos\theta) \\
\vec{v}_A\cdot\vec{V}_1 &= v_A V_{1z}.
\end{aligned}
$$

- Plug these in to the equation

$$-\omega^2\vec{V}_1 + (v_s^2 + v_A^2)(\vec{k}\cdot\vec{V}_1)\vec{k} + (\vec{k}\cdot\vec{v}_A)[(\vec{k}\cdot\vec{v}_A)\vec{V}_1 - (\vec{V}_1\cdot\vec{v}_A)\vec{k} - (\vec{k}\cdot\vec{V}_1)\vec{v}_A] = 0 \;\Rightarrow$$

$$\begin{cases} V_{1x}(k^2v_A^2 + k^2v_s^2\sin^2\theta - \omega^2) + V_{1z}k^2v_s^2\sin\theta\cos\theta = 0 & \text{- x} \\ V_{1y}(k^2v_A^2\cos^2\theta - \omega^2) = 0 & \text{- y} \\ V_{1x}k^2v_s^2\sin\theta\cos\theta + V_{1z}(k^2v_s^2\cos^2\theta - \omega^2) = 0 & \text{- z} \end{cases}$$

Shear Alfvén waves

- Taking $\vec{V}_1 = V_1\vec{e}_y$, gives a solution with the dispersion relation

$(V_{1y} \neq 0)$

$$\frac{\omega^2}{k^2} = v_A^2\cos^2\theta,$$

i.e., the shear Alfvén wave. No propagation across $\vec{B}$: $\omega/k = 0$.

$(\theta = \pi/2)$

- The remaining pair of equations can be written in matrix form as $\mathcal{D}(V_{1x}, V_{1z})^{\mathrm{T}} = 0$ with

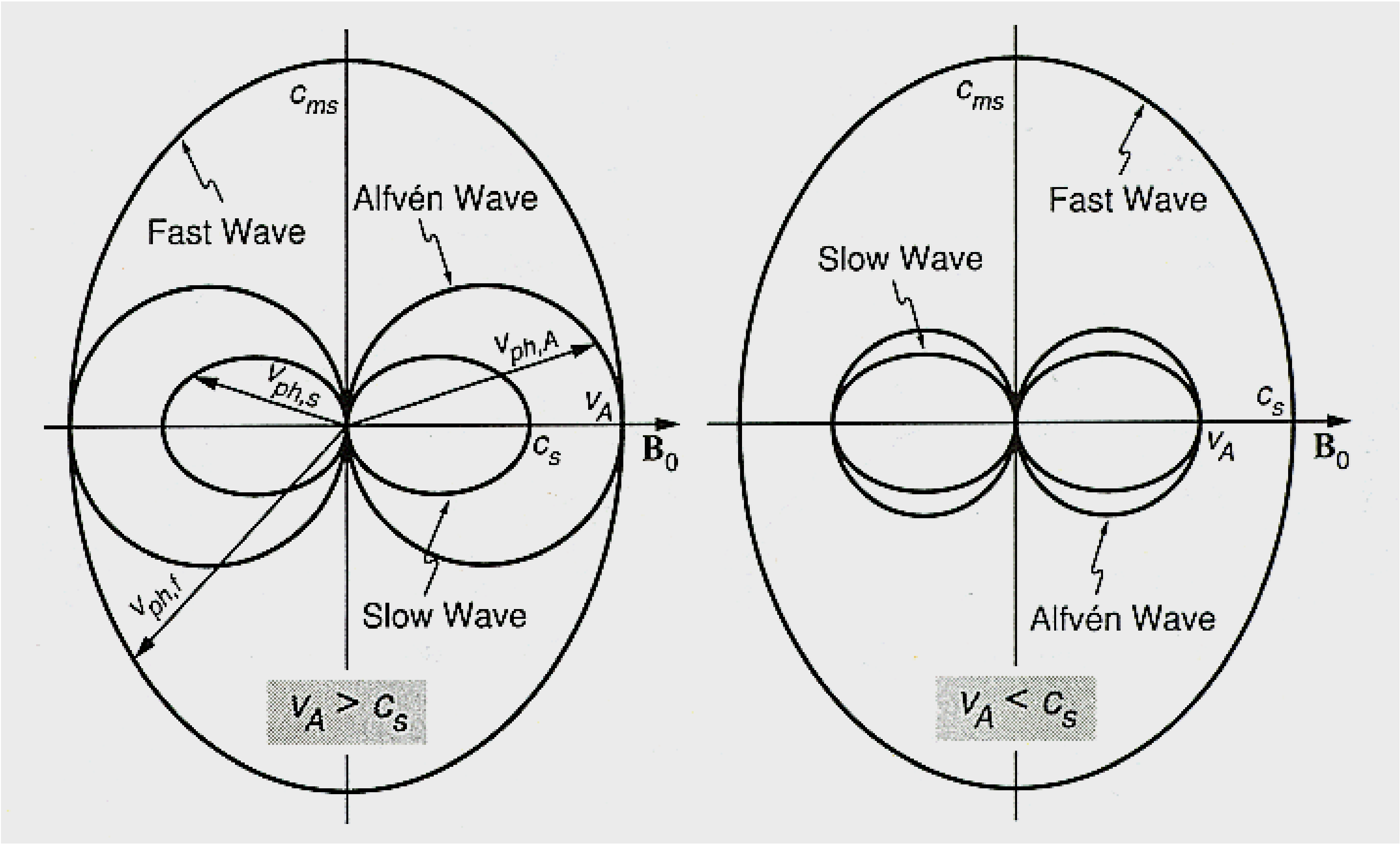
$$\mathcal{D} = \begin{pmatrix} k^2 v_A^2 + k^2 v_s^2 \sin^2\theta - \omega^2 & k^2 v_s^2 \sin\theta\cos\theta \\ k^2 v_s^2 \sin\theta\cos\theta & k^2 v_s^2 \cos^2\theta - \omega^2 \end{pmatrix}$$

- Linear algebra: $\det\mathcal{D} \neq 0 \;\Rightarrow\; \exists\mathcal{D}^{-1} : \; \mathcal{D}^{-1}\mathcal{D} = \mathcal{I} \quad \Rightarrow$
$\mathcal{D}^{-1}\mathcal{D}(v_{1x}, v_{1z})^{\mathrm{T}} = \mathcal{D}^{-1}(0,0)^{\mathrm{T}} \;\Rightarrow\; (v_{1x}, v_{1z}) = (0,0)$.
Demand $\det\mathcal{D} = 0 \;\Rightarrow$ quadratic equation for ω^2/k^2 with the roots

$$\left(\frac{\omega}{k}\right)_{\pm}^2 = \frac{v_s^2 + v_A^2 \pm \sqrt{(v_s^2 + v_A^2)^2 - 4v_s^2 v_A^2 \cos^2\theta}}{2}.$$

These are *fast* (+) and *slow* (−) *Alfvén waves* or *MHD waves*.

Phase-velocity polar diagram of MHD waves



- Note that always $(\omega/k)_- \leq v_A \cos\theta \leq (\omega/k)_+$.

  $\Rightarrow$ shear Alfvén wave sometimes called the *intermediate wave*.

Fast Mode (Cold Plasma)

- In cold plasma, (which is obtained by letting the sound speed V_S =0) magnetic pressure is much greater than kinetic pressure, and also the Alfven velocity is greater than sound speed. The dispersion relation could be simplified as :

$$\omega^2 = k^2 v_A^2$$

- This is the dispersion relation for the compressional-Alfven wave, thus, the fast wave is the compressional-Alfven wave modifed by a non-zero plasma pressure.

- Velocity disturbance is perpendicular to the wave vector.

$$\vec{\mathbf{V}}_1 \cdot \vec{\mathbf{B}}_0 = 0 \qquad \vec{\mathbf{V}}_1 \cdot \vec{\mathbf{k}} = V_1 k_\perp$$

- Electric disturbance $\vec{\mathbf{E}}_1 = -\vec{\mathbf{V}}_1 \times \vec{\mathbf{B}}_0$

- density disturbance $\rho_1 = -(\rho_0 V_1 sin\theta) / v_A$

- Field aligned current $\vec{\mathbf{j}} \cdot \vec{\mathbf{B}}_0 = 0$ prove

- The group and phase velocity is equal and isotropic, and the fast mode propagate in all directions.

$$\frac{\partial}{\partial \mathbf{k}} \omega = v_A \hat{\mathbf{e}}_k$$

- In Warm plasma, dispersion relation is complicated, and group velocity depends on directions and frequency.

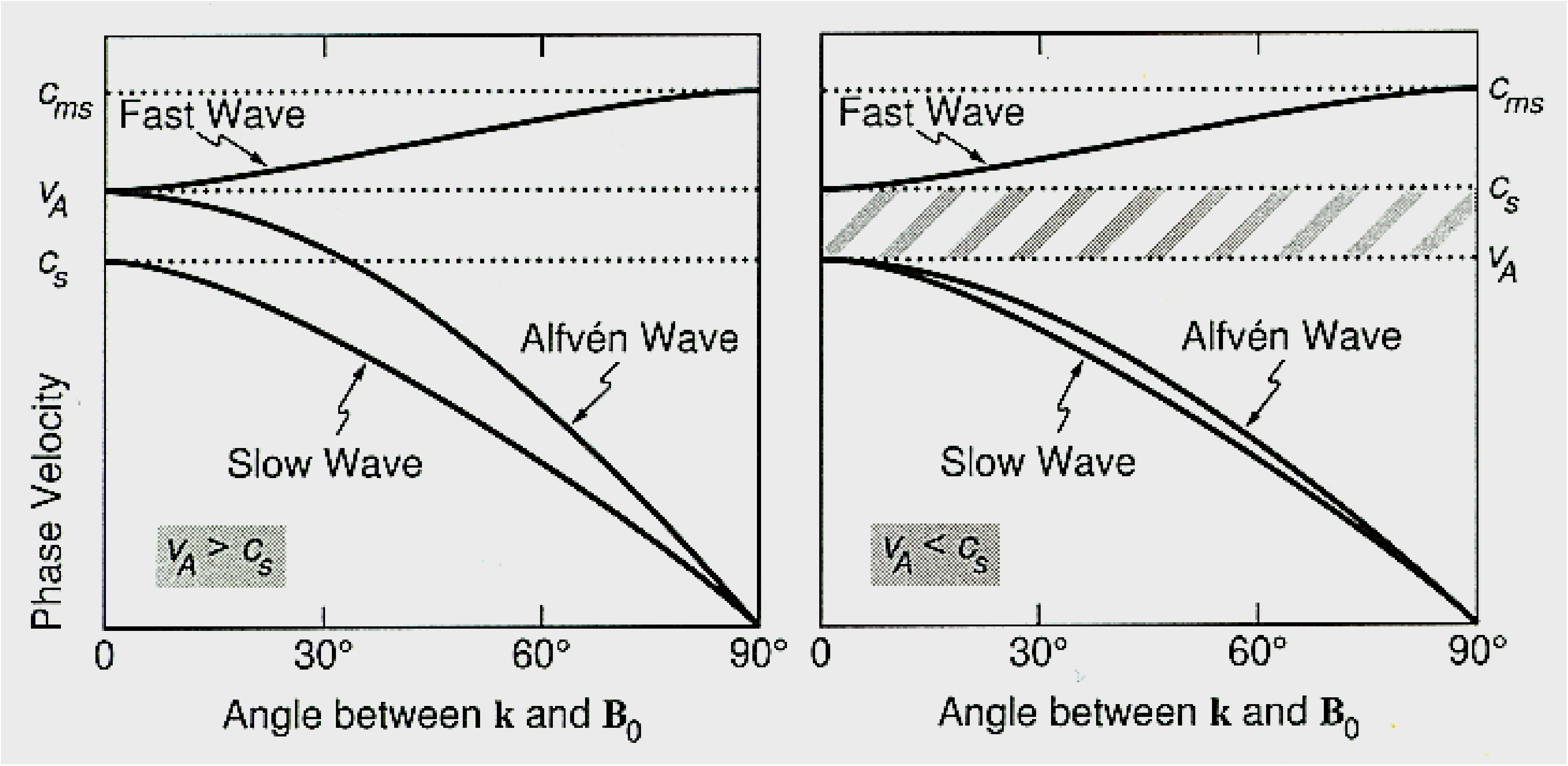
- In cold-plasma limit, which is obtained by letting the sound speed V_S =0. the slow wave ceases to exist (in fact, its phase velocity tends to zero)

- In the limit $V_A >> V_S$, which is appropriate to low-beta, the dispersion relation for the slow wave reduces to

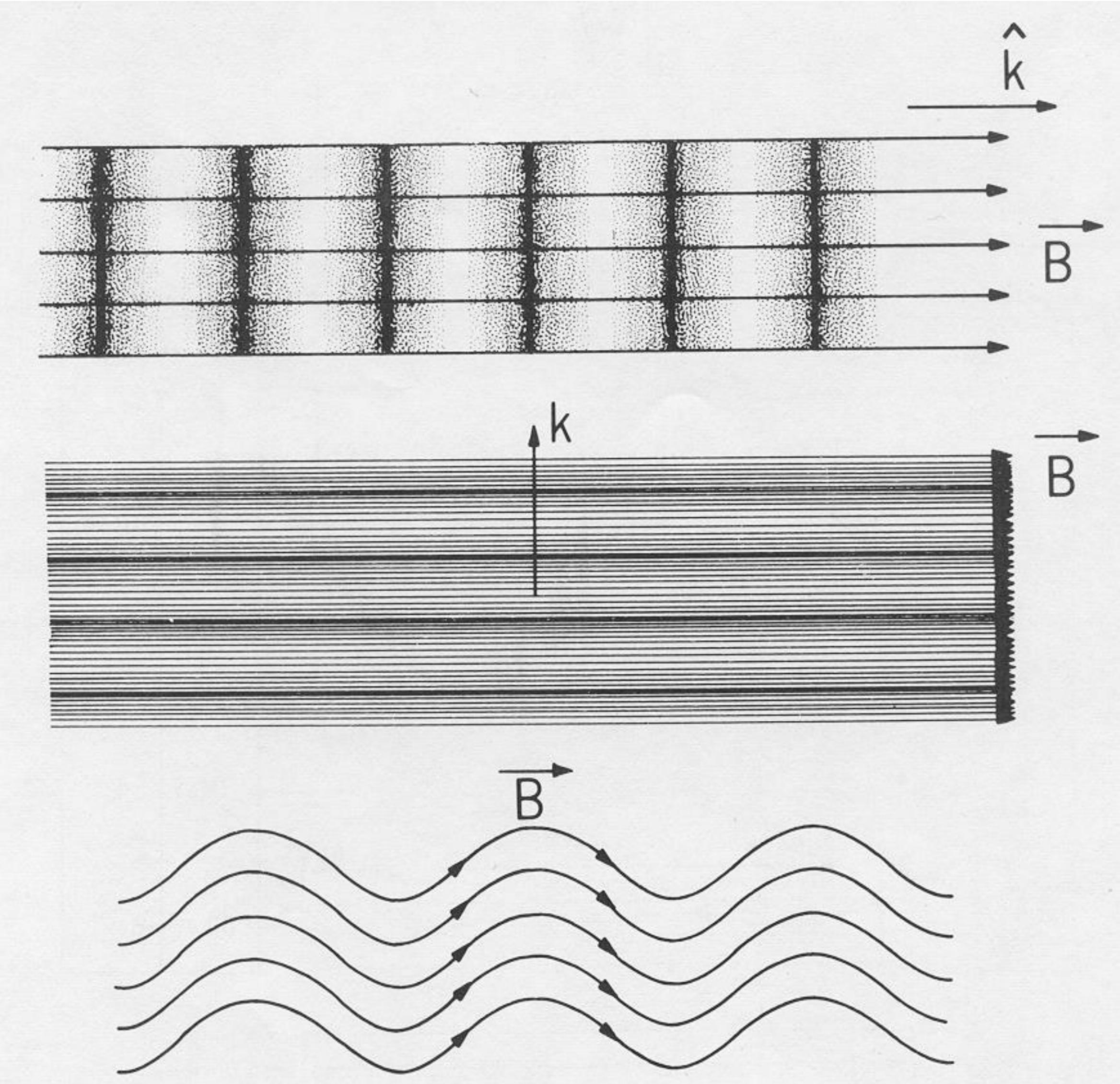
$$\omega \approx k v_s \cos\theta$$

- This is the dispersion relation of a sound wave propagating along magnetic field-lines. Thus, in low-beta plasmas the slow wave is a sound wave modified by the presence of the magnetic field.

Dependence of phase velocity on propagation angle



Magnetohydrodynamic waves



- Magnetosonic waves

compressible

- parallel slow and fast

- perpendicular fast

- Alfvén wave

incompressible

parallel and oblique