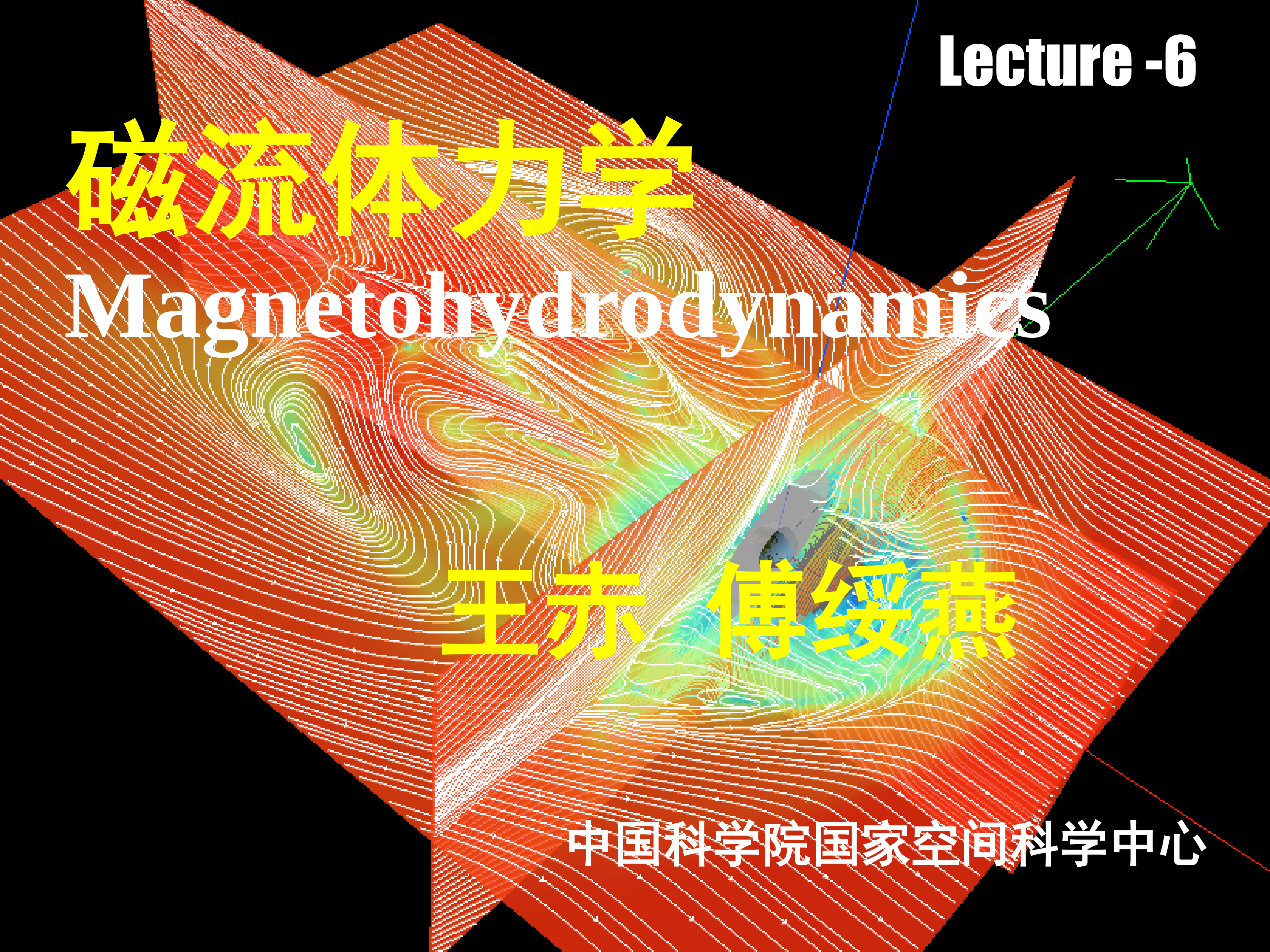
Lecture -6

# 磁流体力学

# Magnetohydrodynamics



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# MHD Instability

# Instabilities in a plasma

*Because of a multitude of free-energy sources in space plasmas, a very large number of instabilities can develop.*

If spatial scale involved is:

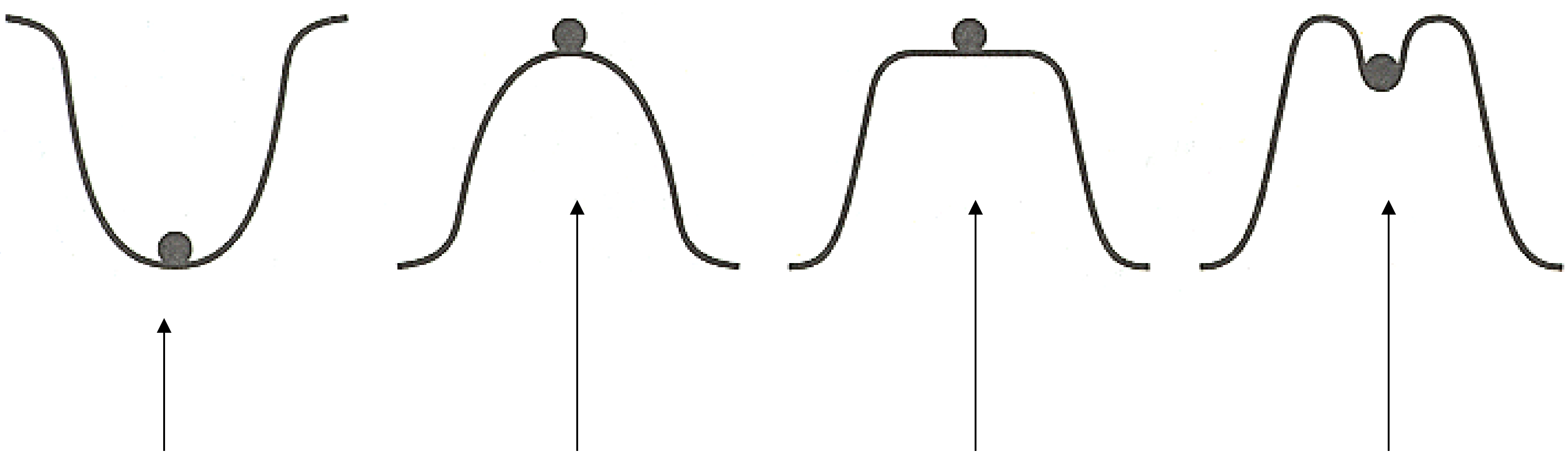
- comparable to macroscopic size (bulk scale of plasma,.....) **->** ***macroinstability*** (affects plasma globally)
- comparable to microscopic scale (gyroradius, inertial length,...) **-> *microinstability*** (affects plasma locally)

Theoretical treatment:

- *macroinstability*, fluid plasma theory
- *microinstability*, kinetic plasma theory

# Concept of instability

*Generation of instability is the general way of redistributing energy which was accumulated in a non-equilibrium state.*



stable    linear unstable    metastable    non-linear unstable

# Methods of Instability Analysis

A method used to study equilibrium problems imagines the system to undergo a small displacement as the result of the application of an arbitrary force. If the force increases the displacement and thereby deforms the system, the system is said to be unstable. If, however, the effects of the force are damped and the system returns to the initial configuration, the system is considered stable.

Normal mode: examine whether the perturbation grows or damps by studying the motion of the particles in the immediate neighborhood.

Energy principle methods: by calculating the energy of the initial and final states.

# Methods of Instability Analysis

Consider two point masses in a one-dimensional potential field V(x) as shown. A small perturbation applied to point A will cause the mass to oscillate about the equilibrium point while perturbation applied to at point B will accelerate the mass away from the equilibrium point. System A is stable and system B is unstable.

Let the coordinate of the equilibrium position be given by $x_0$ and the force *F(x)*. The equation of motion of the mass m at position x, obtained by Taylor expansion about the point $x_0$*:*

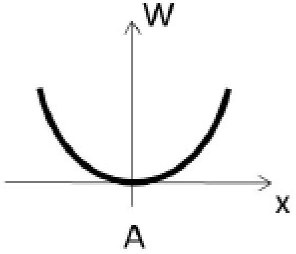
$$m\frac{d^2x}{dt^2} = F(x) = F(x_0) + F'(x_0)(x - x_0) + \cdots$$
$$= F'(x_0)(x - x_0) + \cdots$$

*Where* $x_0$ *is the equilibrium position,* $F(x_0)=0$

# Methods of Instability Analysis

Let $\xi = x - x_0$, and ignored higher-order terms,

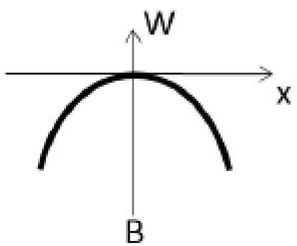
$$m\frac{d^2\xi}{dt^2} = F'(x_0)\xi$$




The solution of this equation:

$$\xi = \xi_0 \exp\{\left[\frac{F'(x_0)}{m}\right]^{1/2} t\} = \xi_0 \exp(i\omega t)$$

$$where\ \omega^2 = -\frac{F'(x_0)}{m}$$

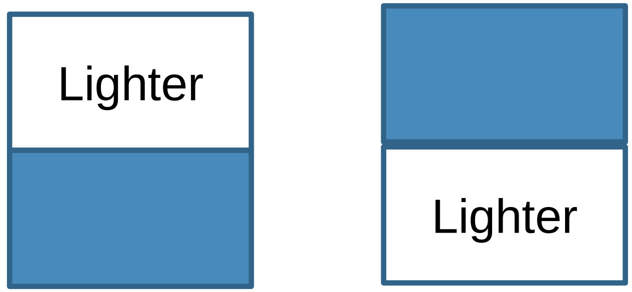



$$F(x)\hat{\mathbf{x}} = -\frac{\partial W}{\partial x}\hat{\mathbf{x}}$$

A: $F'(x_0) < 0$ , $\omega$ real, the solution is oscillatory

B: $F'(x_0) > 0$, the disturbance grows exponentially.

# Interchange Instablity




Consider two vessels in which there are two kinds of fluids in a gravitational field. Let the fluid on the top in one case (left) be lighter than one on the bottom, and let the reverse be true in other case (right). Both systems are initially in equilibrium.

Introduce now a small perturbation in the form of waves, to the interface of two fluids.

Left: the waves will oscillate about the equilibrium and will eventually damp out.

Right: the waves will grow, which will lead to the interchange of the positions of the upper and lower fluids.

The lower energy state is reached by lowering the potential energy.

# Linear instability

The concept of linear instability arises from consideration of a linear wave function. Assume any field (density, field, etc.) denoted by *A*, the fluctuation of which is $\delta A$, that can be Fourier decomposed as

$$\delta A = \sum_{\mathbf{k}} A_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$$

In general the dispersion relation (DR) has complex solutions: $\omega = \omega_r + i\gamma$. For real frequency the disturbances are oscillating waves.

For complex solutions the sign of $\gamma$ decides whether the amplitude *A* growth ($\gamma > 0$) or decays ($\gamma < 0$).

# Two-steream Instability (Buneman Instability)

As a simple example of a streaming instability, consider a uniform plasma the ions are stationary and the electrons have a velocity $v_0$ relative to the ions. Let the plasma be cold $(kT_e = kT_i = 0)$, and let there be no magnetic field $(B_0 = 0)$. We first separate the dependent variables into to two parts: an "equilibrium" part indicated by a subscript 0, and a "perturbation" part indicated by a subscript 1.

$$n_e = n_0 + n_1, \mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_1, \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

The equations of motion for the ions and the electrons are, to first order:

$$Mn_0 \frac{\partial \mathbf{v}_{i1}}{\partial t} = en_0 \mathbf{E}_1$$

$$mn_0\left[\frac{\partial \mathbf{v}_{e1}}{\partial t} + (v_0 \cdot \nabla)\mathbf{v}_{e1}\right] = -en_0 \mathbf{E}_1$$

# Two-steream Instability (Buneman Instability)

We look for electrostatic waves $\mathbf{E}_1 = Ee^{i(kx-\omega t)}\hat{x}$

Where x is the direction of $v_0$

$$-i\omega M n_0 \mathbf{v}_{i1} = e n_0 \mathbf{E}_1 \Rightarrow \mathbf{v}_{i1} = \frac{ie}{M\omega} E\hat{x}$$

$$m n_0(-i\omega + ikv_0)\mathbf{v}_{e1} = -e n_0 \mathbf{E}_1 \Rightarrow \mathbf{v}_{e1} = -\frac{ie}{m} \frac{E\hat{x}}{\omega - kv_0}$$

The velocities are in the x direction, and we may omit the subscript x. The ion equation of continuity yields:

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{i1} = 0 \Rightarrow n_{i1} = \frac{k}{\omega} n_0 v_{i1} = \frac{ien_0 k}{M\omega^2} E$$

Note that the other terms in $\nabla \cdot (nv_i)$ vanish because $v_{oi} = 0$

# Two-steream Instability (Buneman Instability)

The electron equation of continuity is:

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot v_{e1} + (v_0 \cdot \nabla) n_{e1} = 0$$

$$(-i\omega + ikv_0)n_{e1} + ikn_0 v_{e1} = 0$$

$$n_{e1} = \frac{kn_0}{\omega - kv_0} v_{e1} = -\frac{iekn_0}{m(\omega - kv_0)^2} E$$

Since the unstable waves are high-frequency plasma oscillations, we may not use the plasma approximation but use Poisson's equation:

$$\epsilon_0 \nabla \cdot \mathbf{E}_1 = e(n_{i1} - n_{e1})$$

$$ik\epsilon_0 E = e(ien_0 kE)\left[\frac{1}{M\omega^2} + \frac{1}{(\omega - kv_0)^2}\right.$$

# Two-steream Instability (Buneman Instability)

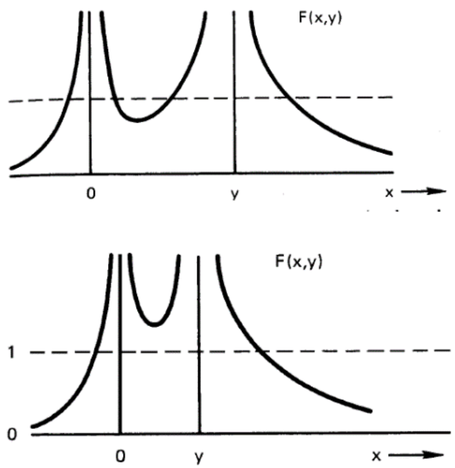
The dispersion relation is found upon dividing by $ik\epsilon_0 E$

$$1 = \omega_p^2 \left[ \frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2} \right]$$

Let us see if oscillations with real k are stable or unstable. If all the roots $\omega_j$ are real, each root would indicate a possible oscillation.

If some of the roots are complex, they will occur in complex conjugate pairs.  Positive Im(ω) indicates an exponentially growing wave; negative Im(ω) indicates a damped wave. Since the roots occur in conjugate pairs, one of these will always be unstable unless all the roots rare real.

The dispersion relation can be analyzed without actually solving the fourth-order equation. Let us define




$$x \equiv \frac{\omega}{\omega_p};\ y = \frac{kv_0}{\omega_p}$$

$$1 = \frac{m/M}{x^2} + \frac{1}{(x-y)^2} = F(x,y)$$

For any given value of y, we can plot F(x,y) as a function of x. This function will have singularities at x = 0 and x =y. In the example of (a), there are four intersection's, so there are four real roots (stable). However if we choose a smaller value of y, there are only two real roots. The other two roots must be complex, and one of them must correspond to an unstable wave. Thus, for sufficiently small $kv_0$, the plasma is unstable.

# Growth rate

$$1 = \omega_p^2 \left[ \frac{m/M}{\omega^2} + \frac{1}{(\omega - kv_0)^2} \right]$$

For 0$^{th}$ approximation, m/M $\rightarrow$ 0, then:

$$kv_0 - \omega \sim \omega_p$$

If m/M cannot be ignored, the above equation must be examined in its entirety. We can assume $kv_0 - \omega = \omega_p - \delta\omega$ , $(\delta\omega \ll \omega)$then

$$1 - \frac{\omega_p^2}{(\omega_p - \delta\omega)^2} = \frac{\omega_p^2 (\frac{m}{M})}{(kv_0 - \omega_p + \delta\omega)^2}$$

The low frequency case $\omega_{pi} \ll \omega \ll \omega_{p(e)}$ has wavenumber $k \approx \omega_p / v_0$, this permits to write the dispersion relation as

$$(\delta\omega)^3 \sim (-\frac{m}{M})\omega_p^3$$

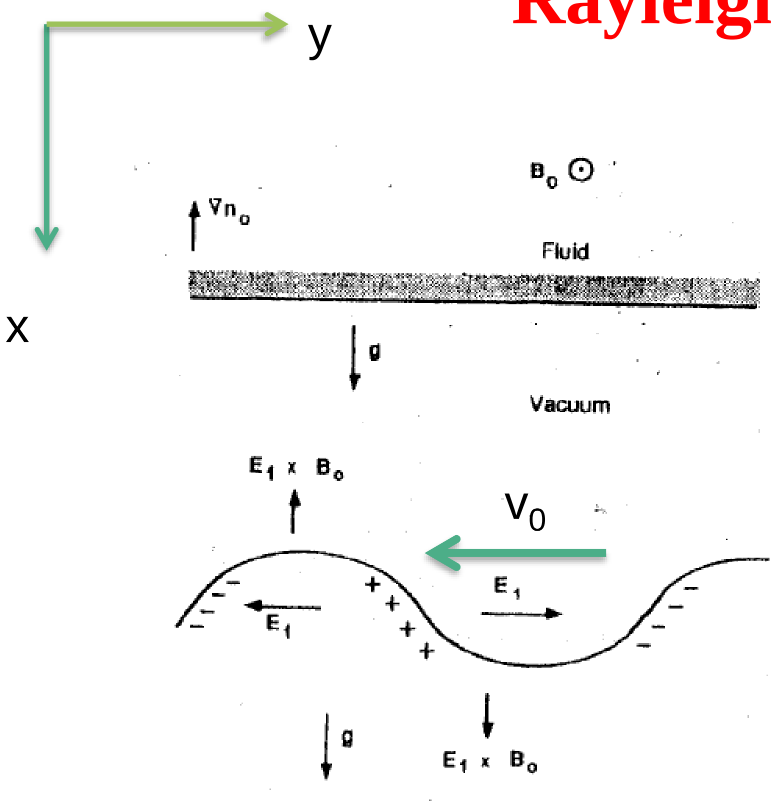
Which has one real negative frequency stable solution of no interest and two conjugate complex solutions which can be found when putting $\delta\omega = \omega_r + i\gamma$ and separating into real and imaginary parts

$$\omega_r\left(\omega_r^2 - 3\gamma^2\right) = -m\omega_p^3 \text{ and } \gamma^2 = 3\,\omega_r^2$$

Which yields the oscillating wave of frequency $\omega_r = \omega_B$ and growth rate $\gamma = \gamma_B$ with

$$\omega_B = \omega_p\left(\frac{m}{16M}\right)^{1/3} \quad \text{and} \quad \gamma_B = 3^{1/2}\,\omega_B$$

# Rayleigh-Taylor instability




In a plasma, a R-T instablitiy can occur because the magntec field acts as a light fluid supporting a heavy fluid (the plamsa). To treat the simplest case, coonsider a plamsa boundary lying in the y-z plane. Let $\mathbf{B}_0$ be in the z-direction. We assume the plasma β is low so that we can let kTe = kTi = 0. This implies that there is no diamagnetic current (due to $\nabla n$ )

In the equibrium state, the ions obey the equation:

$$Mn_0(v_0 \cdot \nabla)v_0 = qn\mathbf{v}_0 \times \mathbf{B}_0 + Mn_0\mathbf{g}$$

If g is constant, $v_0$ will be also, and $(v_0 \cdot \nabla)v_0$ vanishes. Taking the cross product of the above equation with $\mathbf{B}_0$ ,

$$\mathbf{v}_0 = \frac{M}{q}\frac{\mathbf{g} \times \mathbf{B}_0}{B_0^2} = -\frac{g}{\Omega_c}\hat{y}$$

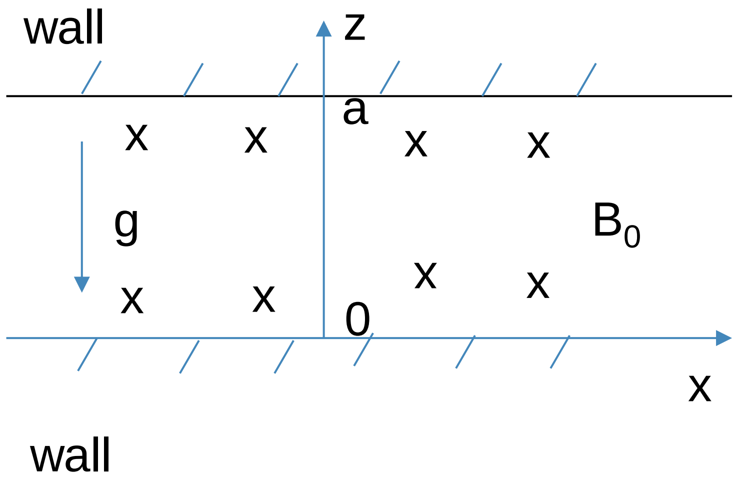
$$(\boldsymbol{v}_0 \times \boldsymbol{B}_0) \times \boldsymbol{B}_0 = B_0(\boldsymbol{v}_0 \cdot \boldsymbol{B}_0) - \boldsymbol{v}_0(B_0 \cdot \boldsymbol{B}_0) = -\boldsymbol{v}_0 B_0^{\ 2}$$

which is just the guiding center drift of ions acted on by the gravitational force. Here $\Omega_c = qB/M$ is the on Larmor frequency.

We can obtain a similar equation for the electrons that drift in the opposite direction. However, in the limit m/M ~ 0, the electron contribution can be ignored.

Introduce now a small disturbance so that the boundary becomes ripped. Because the $\boldsymbol{g} \times \boldsymbol{B}$ drift is mass dependent, the ions will drift faster than the electrons, hence it can be easily deduced that the $v_0$ of the ions over the rippled surface will cause the charges to build up as shown. This charge separation produces an electric field $E_1$ and the ripples, $\boldsymbol{E}_1 \times \boldsymbol{B}$ is in the x-direction the minimum and in the –x direction at the peaks. The amplitude of the ripple will thus grow larger and the boundary becomes unstable.

We will carry out the analysis using the normal mode method to investigate the R-H instability.




The basic geometry is shown in the figure. We assume that the plasma is bounded perfectly conducting walls at z=0, a, that the gravitational acceleration, **g**, is directed downward, and that magnetic field, $\mathbf{B}_0$, is in the +y direction (i.e. into the paper)

Assume that the equilibrium mass density profile is given by exponential $\rho_{m0} = \rho_0 \exp(-\frac{z}{Hs})$, where $\rho_0$ is the density at z = 0, and Hs is a constant that is called the scale height in the ionospheric literature. If Hs>0, the equilibrium density decreases with the increasing height. If Hs < 0, the equilibrium density increases with increasing height.

In the presence of gravity, the momentum equation must be modified to include the gravitational force， $\rho_m \boldsymbol{g}$, so that,

$$\rho_m \frac{\boldsymbol{dU}}{\boldsymbol{dt}} = \frac{\mathbf{1}}{\boldsymbol{\mu_0}}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} - \boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0} + P\right) + \boldsymbol{\rho_m}\mathbf{g}$$

Linearizing this equation with respect to small perturbations around the zero-order equilibrium gives

$$\rho_{m0} \frac{\boldsymbol{dU}}{\boldsymbol{dt}} = \frac{\mathbf{1}}{\boldsymbol{\mu_0}}(\boldsymbol{B_0}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B_0} - \boldsymbol{\nabla}\left(\frac{\boldsymbol{B}_0\cdot\boldsymbol{B}}{\mu_0} + P\right) + \boldsymbol{\rho_m}\mathbf{g}$$

Where, we have suppressed the subscript 1 on the first order terms. Next consider a plane wave perturbation on the form $\exp(ikx + i\omega t)$, which assumes no spatial variation along y, and hence the equilibrium magnetic filed suffers no spatial modulation along y. The first two terms on the right above vanish. It is convenient to eliminate the third term by taking curl, which gives:

$$\boldsymbol{\nabla} \times (\rho_{m0} \frac{\boldsymbol{dU}}{\boldsymbol{dt}}) = \boldsymbol{\nabla} \times \rho_{\boldsymbol{m}} \mathbf{g}$$

The y component of the above equation is given by

$$\omega \left[ ik\rho_{m0}\widetilde{U_z} - \frac{\partial}{\partial z}(\rho_{m0}\widetilde{U_x}) \right] = kg\widetilde{\rho_m} \quad (1)$$

For simplicity, assume the perturbed velocity is incompressible.

$$\nabla \cdot \boldsymbol{U} = 0$$

For assumed plane wave perturbation,

$$ik\,\widetilde{U_x} + \frac{\partial \widetilde{U_z}}{\partial z} = 0 \quad \text{or } \widetilde{U_x} = \frac{i}{k}\frac{\partial \widetilde{U_z}}{\partial z} \qquad (2)$$

Since $\nabla \cdot \boldsymbol{U}=0$, the linearized mass continuity equation simplifies to

$$\frac{\partial \rho_m}{\partial t} + \boldsymbol{U} \cdot \nabla \rho_{m0} = 0$$

Which for the assumed plane wave perturbation yields

$$i\omega\widetilde{\rho_m} + \widetilde{U_z}\frac{\partial\rho_{m0}}{\partial z} = 0 \quad \text{or} \quad \widetilde{\rho_m} = \frac{i\widetilde{U_z}}{\omega}\frac{\partial\rho_{m0}}{\partial z} \qquad (3)$$

It is straightforward to eliminate $\widetilde{U_x}$ and $\widetilde{\rho_m}$ from equation (1) using (2) and (3), which gives

$$\frac{1}{\rho_{m0}}\frac{\partial}{\partial z}\left(\rho_{m0}\frac{\partial\widetilde{U_z}}{\partial z}\right) = k^2\left(1 - \frac{g}{H_s\omega^2}\right)\widetilde{U_z}$$

Where $^1/_{H_s} = -\left(\frac{1}{\rho_{m0}}\right)\frac{\partial\rho_{m0}}{\partial z}$

The above equation is a second-order differential equation for $\widetilde{U_z}$, to be solved using the boundary conditions $\widetilde{U_z} = 0$ at z=0 and a.

To solve the above equation, we make the substitution:

$\widetilde{U_z}(z) = f(z)\exp\left(\frac{z}{2Hs}\right)$, which gives

$$\frac{d^2 f}{dz^2} + \alpha^2 f = 0$$

Where $\alpha^2 = k^2 \left( \frac{g}{H_s \omega^2} - 1 \right) - \frac{1}{4H_s^2}$

Since f(0) = f(a) = 0, we obtain the eigenfunctions $f_n(z) = f_0 \sin \frac{n\pi z}{a}$, n = 1, 2, 3, …, where $f_0$ is a constant, and the eigenvalue relation:

$$\frac{n^2 \pi^2}{a^2} = k^2 \left( \frac{g}{H_s \omega_n^2} - 1 \right) - \frac{1}{4H_s^2}$$

The above equation can be solved for the frequencies of normal modes which are give by

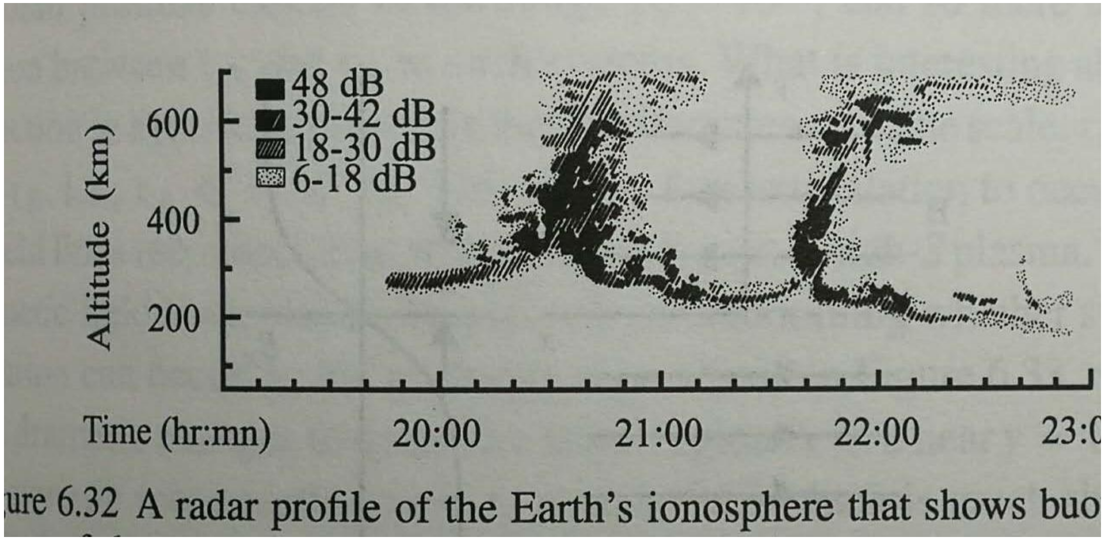
$$\omega_n^2 = \left(\frac{g}{H_s}\right)\frac{4k^2a^2H_s^2}{a^2 + 4H_s^2(k^2a^2 + n^2\pi^2)}$$

Note that

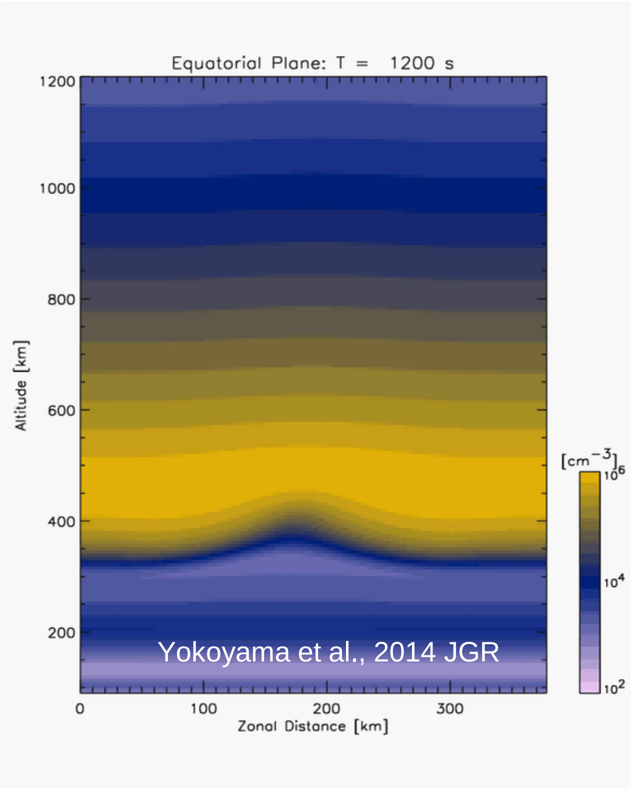
1. If Hs > 0, the frequency $\omega_n$, of the nth mode is purely real and the system is stable. On the other hand, if Hs<0, the frequency is purely imaginary, and the system is unstable, which means $\partial\rho_{m0}/\partial z > 0$, having opposite sign of **g**. This is just the statement that the light fluid is supporting the heavy fluid.
2. The largest growth rate, $\sqrt{g/H_s}$, occurs in the limit $k \to \infty$, and the smallest growth rate occurs in the limit $k \to 0$
3. For fixed k, the growth rate decreases with increasing mode number n.

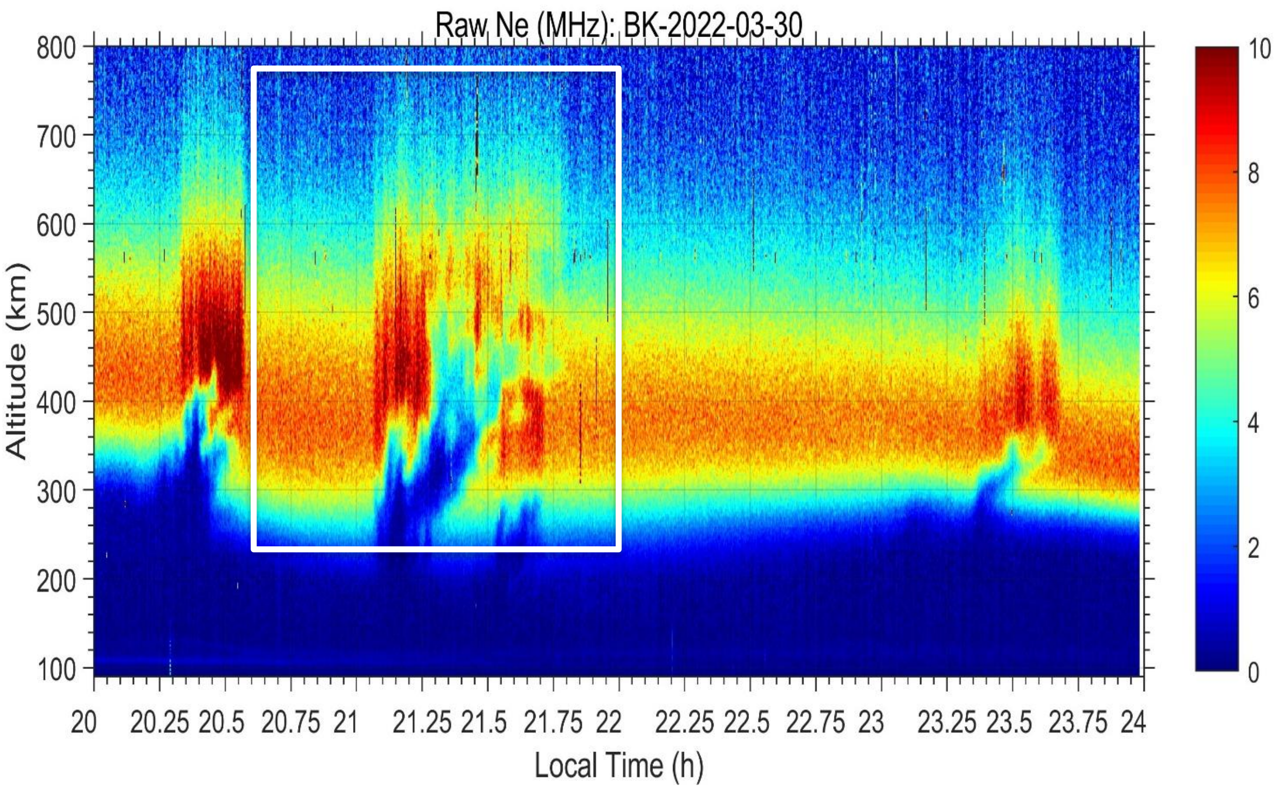
# RT instablity occurs in the Ionosphere

A good example of the RT instability occurs in the ionosphere near the magnetic equator, where the magnetic field is nearly horizontal. Under certain conditions, "bubbles' of low density plasma from near the base of the ionosphere rise upwards into the ionosphere, causing plume-like disturbances, that can be detected by ground-based radars.



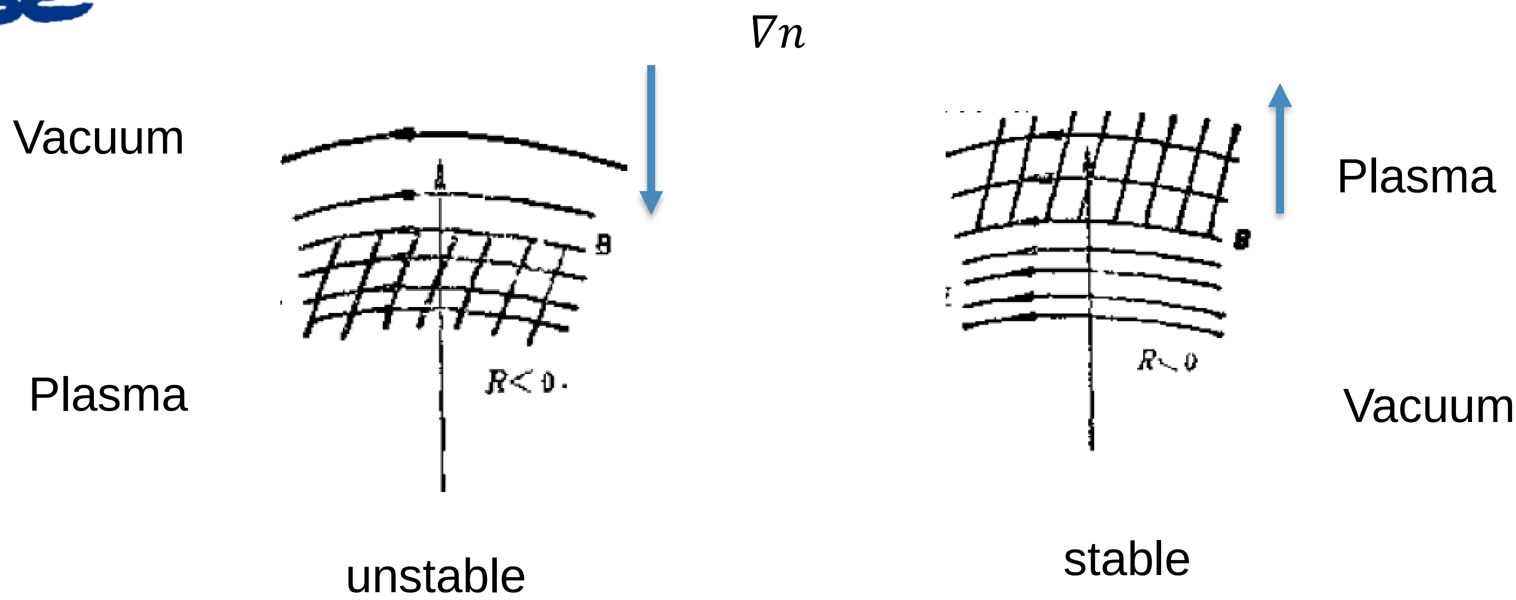

Basu and Kelley, 1979




- ☐ **left：Simulation**
- ☐ **Right: SYISR observation**

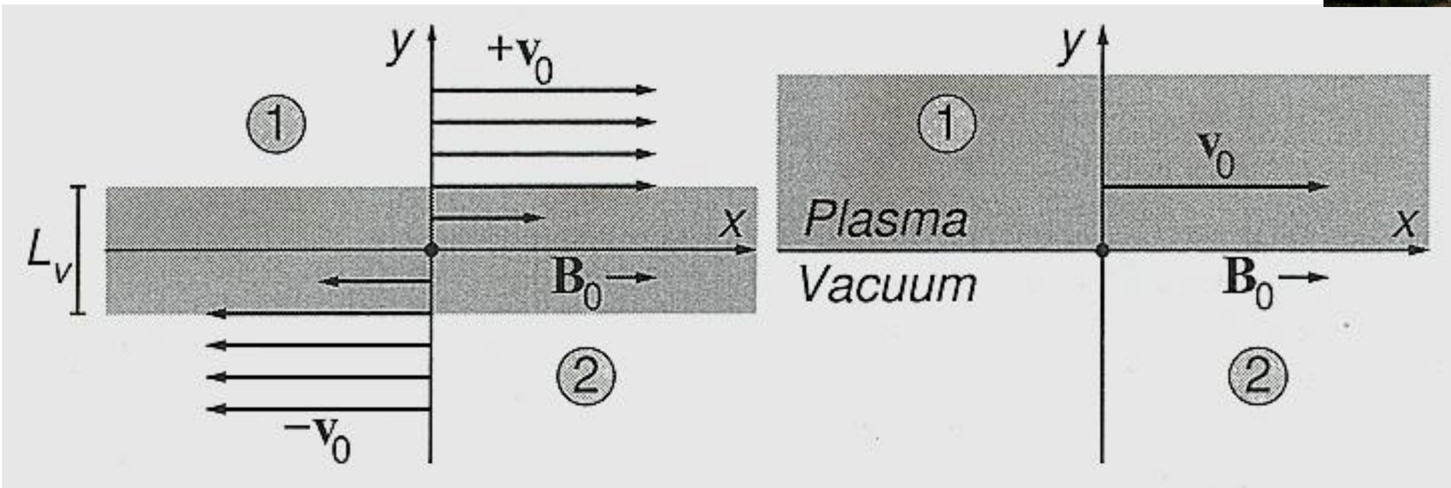
Credit: Yue X.A.




$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(mv_{/\!/}^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}$$

Since g can be used to model the effects of magnetic field curvature, we see from this that stability depends on the sign of the curvature. Configurations with field lines bending in toward the plasma tend to be stabilizing, and vice versa.

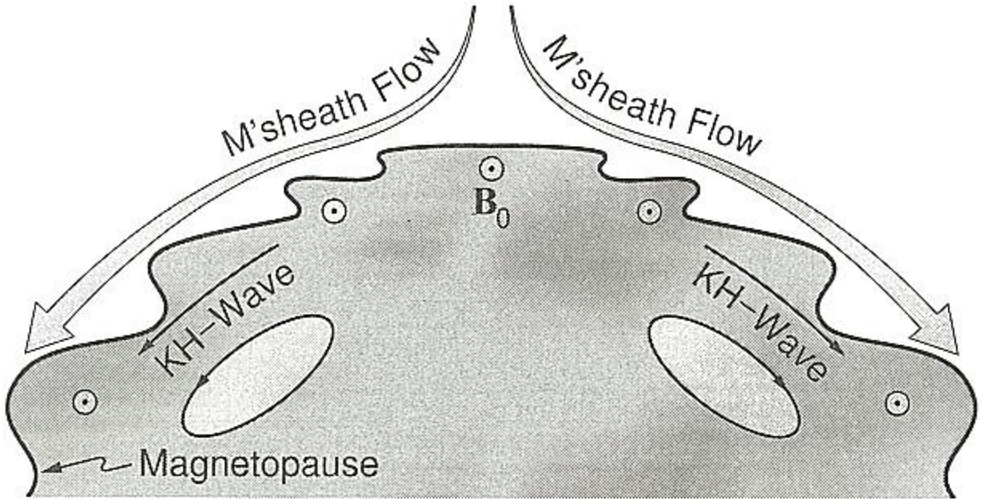
# Kelvin-Helmholtz instability




- ***Shear flow at magnetised plasma boundary may cause ripples on the surface that can grow.........***
- ***The rigidity of the field provides the dominat restoring***

Linear perturbation analysis in both regions shows that ***incompressible waves confined to the interface*** can be excited

$$\frac{1}{n_{02}\left[\omega^2-(\mathbf{k}\cdot\mathbf{v}_{A2})^2\right]}+\frac{1}{n_{01}\left[(\omega-\mathbf{k}\cdot\mathbf{v}_0)^2-(\mathbf{k}\cdot\mathbf{v}_{A1})^2\right]}=0$$




*Excitation of geomagnetic pulsations!*

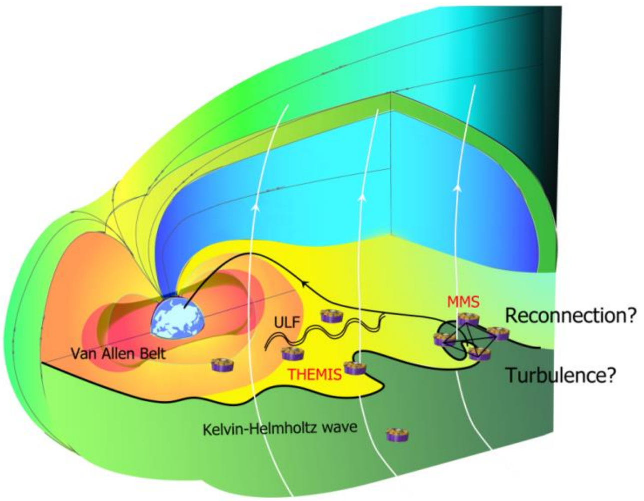
The in ω quadratic dispersion relation yields an unstable solution given by:

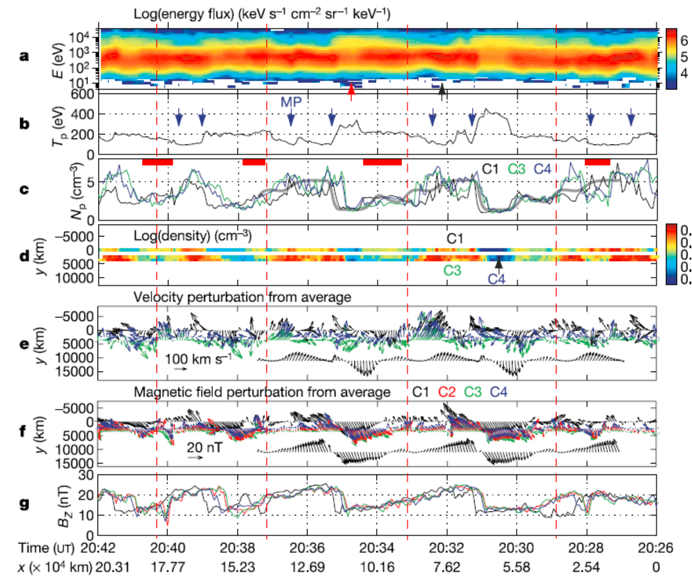
$$\omega_{kh} = \frac{n_{01}\mathbf{k}\cdot\mathbf{v}_0}{n_{01}+n_{02}}$$

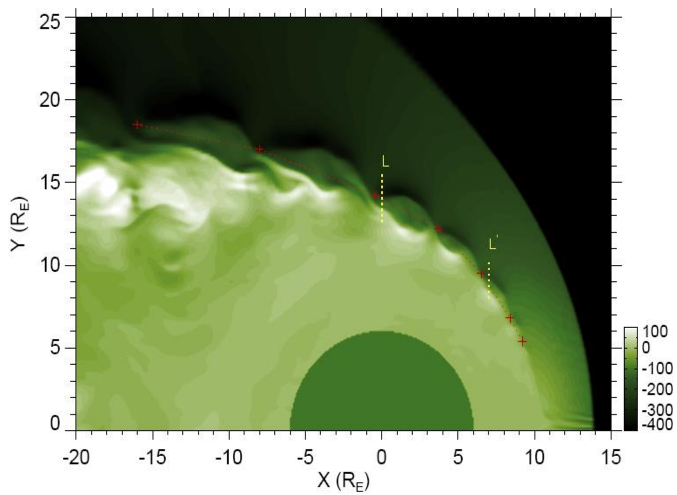
corresponding to the appearance of a complex conjugate root if the streaming is large enough, i.e. if the subsequent inequality is fulfilled:

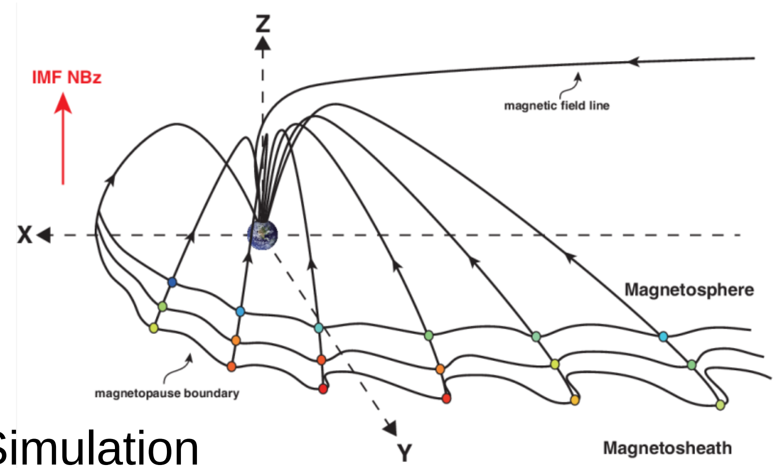
$$(\mathbf{k}\cdot\mathbf{v}_0)^2 > \frac{n_{01}+n_{02}}{n_{01}n_{02}}\left[n_{01}(\mathbf{k}\cdot\mathbf{v}_{A1})^2 + n_{02}(\mathbf{k}\cdot\mathbf{v}_{A2})^2\right]$$

# KH instability at the magneotpause



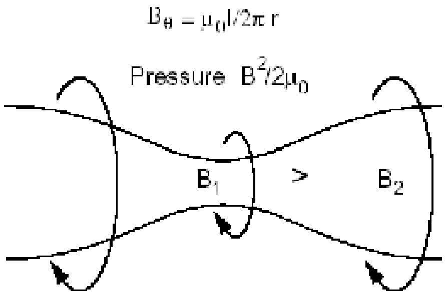




3D Global MHD Simulation

# Sausage Instability

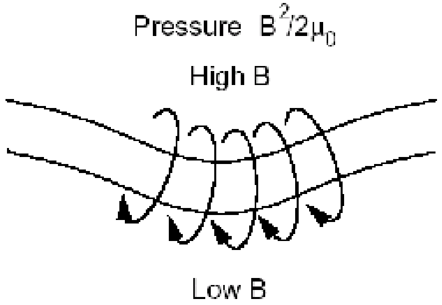



Suppose that the equilibrium state of the pinched plasma column, is disturbed by a wave-like perturbation.

We shall consider that the plasma is constricted in some locations and expanded at others, in such a way that its volume does not change. So, the pressure plasma is left unchanged.

In view of the 1/r radial dependence of the azimuthal magnetic field, at the location where the radius has decreased, the magnetic pressure will be larger than the plasma pressure, and will force the surface radially inwards, thus enhancing the constriction. (m = 0 sausage instability)

# Kink Instability



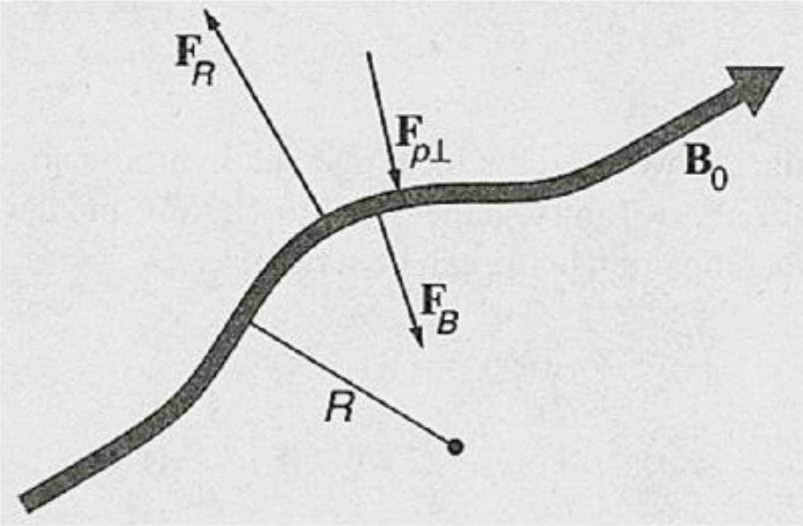

The kink distortion consists of a perturbation in the form of a bend or kink in the column.

If the column develops a kink, the increased pressure and tension on the high B side increases the bending. This is known as the m =1 kink instability.

Instabilities can be prevented by adding a longitudinal magnetic field Bz to "stiffen" the plasma. The $B^2/_{2\mu_0}$ pressure resists the m = 0 mode and the tension $B^2/_{\mu_0}$ counters the bending.

# Firehose Instability

Whenever the flux tube is slightly bent, the plasma exerts an outward centrifugal force (curvature radius, R), that tends to enhance the initial bending. The gradient force due to magnetic stresses and thermal pressure resist the centrifugal force. In force equilibrium:




$$\frac{m_i n_0 v_{th\|}^2}{R} = \frac{p_\perp}{R} + \frac{B_0^2}{\mu_0 R}$$

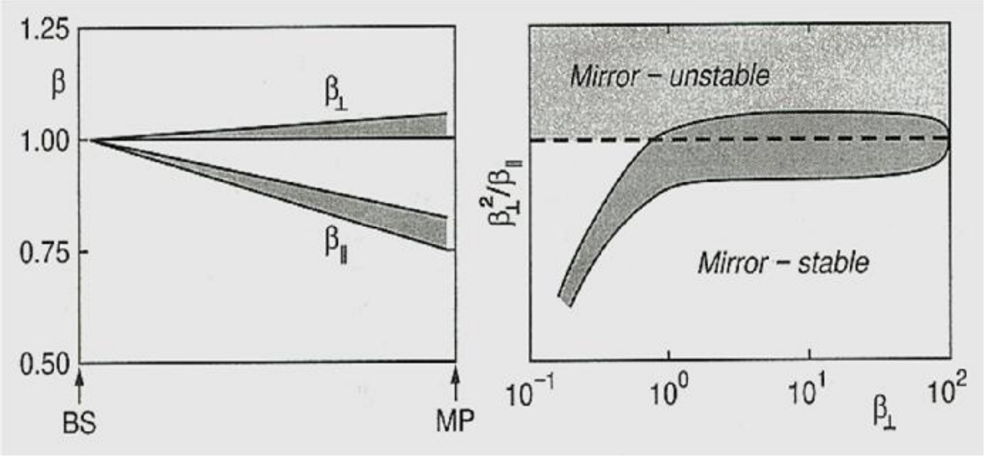
The resulting instability condition (anisotropic pressure instability) for breaking equilibrium is:

$$p_\| > p_\perp + B_0^2/\mu_0$$

# Mirror Instability

This long-wavelength compressive slow-mode instability requires consideration of particle motion parallel and perpendicular to the field.

Occurs in the Earth's dayside magnetosheath, where the shocked solar is heated adiabatically in the perpendicular direction, while the field-aligned outflow cools the plasma in the parallel direction.



# Mirror Instability

The particles stream into the mirror during instability, become trapped there and oscillate between mirror points. Density and field out of phase***, slow mode wave!***