Basic Space Plasmas Physics
Assignment 4

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(Solve three out of the four problems! If time permits, you can solve all.)

1 Proof

1.

$$
\begin{aligned}
&-\frac{1}{2}v^2\left[\nabla\cdot(\rho\vec{v})\right]-\rho\vec{v}\cdot[(\vec{v}\cdot\nabla)\vec{v}]+\nabla\cdot(\frac{1}{2}\rho v^2\vec{v})\\
&=\rho\vec{v}\cdot\nabla(\frac{1}{2}v^2)-\rho\vec{v}\cdot[(\vec{v}\cdot\nabla)\vec{v}]\\
&=\rho\vec{v}\cdot[\vec{v}\times(\nabla\times\vec{v})]\\
&=0
\end{aligned}
\tag{1.1}
$$

2.

$$
\begin{aligned}
&\vec{v}\cdot(\nabla\cdot\Pi)-\nabla\cdot[\vec{v}\cdot\Pi]\\
&=v_i\frac{\partial}{\partial x_j}\Pi_{ji}-\frac{\partial}{\partial x_j}(v_i\Pi_{ij})\\
&=v_i\frac{\partial}{\partial x_j}\Pi_{ji}-v_i\frac{\partial}{\partial x_j}\Pi_{ij}-\Pi_{ij}\frac{\partial}{\partial x_j}v_i\\
&=-\Pi_{ij}\frac{\partial}{\partial x_j}v_i\quad(\Pi\text{ is a symmetric matrix})\\
&=-\nabla\vec{v}:\Pi\quad(or-\Pi:\nabla\vec{v})
\end{aligned}
\tag{1.2}
$$

3.

$$
\begin{aligned}
&\vec{H}\cdot\left[\nabla\times(\vec{v}\times\vec{H})\right]+\vec{v}\cdot\left[(\nabla\times\vec{H})\times\vec{H}\right]+\nabla\cdot\left[\vec{H}\times(\vec{v}\times\vec{H})\right]\\
&=\vec{H}\cdot\left[\nabla\times(\vec{v}\times\vec{H})\right]+\vec{v}\cdot\left[(\nabla\times\vec{H})\times\vec{H}\right]+(\nabla\times\vec{H})\cdot(\vec{v}\times\vec{H})-\vec{H}\cdot\left[\nabla\times(\vec{v}\times\vec{H})\right]\\
&=0
\end{aligned}
\tag{1.3}
$$

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4.

$$
\begin{aligned}
&\vec{H}\cdot\nabla^2\vec{H}-\nabla\cdot\left[\vec{H}\times(\nabla\times\vec{H})\right]\\
&=\vec{H}\cdot\nabla^2\vec{H}-(\nabla\times\vec{H})\cdot(\nabla\times\vec{H})+\vec{H}\cdot\left[\nabla\times(\nabla\times\vec{H})\right]\\
&=\vec{H}\cdot\nabla^2\vec{H}-\left|\nabla\times\vec{H}\right|^2+\vec{H}\cdot\left[\nabla(\nabla\cdot\vec{H})-\nabla^2\vec{H}\right]\\
&=-\left|\nabla\times\vec{H}\right|^2
\end{aligned}
\tag{1.4}
$$

2 Magnetohydrodynamics Energy Equation

The continuity equation is given by

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0 \tag{2.1}$$

where ρ is the mass density and $\vec{v}$ is the flow velocity.

The momentum equation (equation of motion) is given by

$$\rho(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v})=\frac{\mu}{4\pi}(\nabla\times\vec{H})\times\vec{H}-\nabla p+\nabla\cdot\Pi \tag{2.2}$$

where $\vec{H}$ is magnetic field intensity, μ is a material dependent parameter called the permeability, p is the scalar pressure, and Π is the viscous stress tensor.

$$\Pi_{ik}=\zeta(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\nabla\cdot\vec{v})+\zeta^{'}\delta_{ik}\nabla\cdot\vec{v} \tag{2.3}$$

The energy equation (conservation of energy) is given by

$$\frac{\partial W}{\partial t}+\nabla\cdot\vec{q}=0 \tag{2.4}$$

where W is total energy per unit volume and $\vec{q}$ is the energy flux density through the boundary of the fluid element. The total energy is the sum of the kinetic, magnetic, and internal energies:

$$W=\frac{1}{2}\rho v^2+\frac{\mu}{8\pi}H^2+\rho e \tag{2.5}$$

where e is the internal energy per unit mass. The energy flux density is given by

$$
\begin{aligned}
\vec{q} &= \rho\vec{v}(\frac{1}{2}v^2+e+\frac{p}{\rho})+\frac{\mu}{4\pi}\vec{H}\times(\vec{v}\times\vec{H})\\
&\quad -\frac{1}{\sigma}(\frac{c}{4\pi})^2\vec{H}\times(\nabla\times\vec{H})-\vec{v}\cdot\Pi-\chi\nabla T
\end{aligned}
\tag{2.6}
$$

where σ is the electrical conductivity, χ is the thermal conductivity, and T is the temperature.

Derive the non-conservation form of the energy equation as the following, which only contains the time derivative of e.

$$\rho\frac{\partial e}{\partial t}+\rho\vec{v}\cdot\nabla e=-p\nabla\cdot\vec{v}+\nabla\vec{v}:\Pi+\nabla\cdot(\chi\nabla T)+\frac{1}{\sigma}J^2 \tag{2.7}$$

Hint: The magnetohydrodynamics equations are all shown in Gaussian electromagnetic units system. The equations of magnetic field are given by

$$\frac{\partial\vec{H}}{\partial t} = \nabla\times(\vec{v}\times\vec{H})+\frac{c^2}{4\pi\mu\sigma}\nabla^2\vec{H} \tag{2.8}$$

$$\nabla\cdot\vec{H} = 0 \tag{2.9}$$

3 Frozen Flux Theorem

Under ideal MHD assumption, the magnetic field equations (SI units) reduce to

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{3.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{3.2}$$

where $\vec{B}$ is the magnetic flux density. Consider a closed curve C within the fluid, and let every point on the curve be moving with the local fluid velocity. We say that C is co-moving with the fluid, in the Lagrangian sense. Let S be a surface boundary by C, then we can obtain the surface integral:

$$\iint \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S} - \oint (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{C} = 0 \tag{3.3}$$

As $S_1(t)$ moves in a infinite short time $\mathrm{d}t$, $S_1(t)$, $S_2(t+\mathrm{d}t)$, and swept lateral area S_3 form a closed surface. This is shown in Figure 1 at time $t + \mathrm{d}t$. For $\nabla \cdot \vec{B} = 0$, we can obtain that

$$\iint_{S_2} \vec{B}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{S} - \iint_{S_1} \vec{B}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{S} + \iint_{S_3} \vec{B}(t+\mathrm{d}t) \cdot \mathrm{d}\vec{S} = 0 \tag{3.4}$$

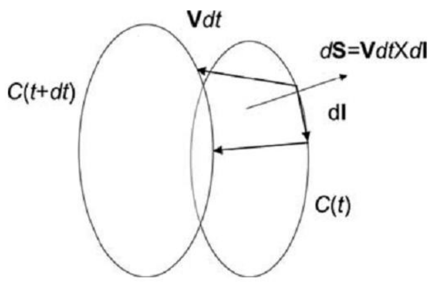

Figure 1: A surface moving with the fluid

Let $\mathrm{d}t \to 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{B} \cdot \mathrm{d}\vec{S} - \iint_S \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S} + \oint_C (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{C} = 0 \tag{3.5}$$

With (3.3), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \vec{B} \cdot \mathrm{d}\vec{S} = 0 \tag{3.6}$$

We conclude that in ideal MHD, the magnetic flux through any co-moving closed circuit remains constant. This important result is called the frozen flux condition.

Consummate all the derivation.

4 Magnetic Diffusion

A unidirectional magnetic field $\vec{B} = B(x,t)\vec{e}_y$ has the initial form

$$B(x,0) = \begin{cases} +B_0, & x > 0 \\ -B_0, & x < 0 \end{cases} \tag{4.1}$$

If the magnetic Reynolds number is very small, according to the induction equation, the evolution of the magnetic field can be described as

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2} \tag{4.2}$$

where η is the magnetic viscous coefficient. The solution is

$$B(x,t) = B_0 \mathrm{erf}(\xi) \tag{4.3}$$

where $\xi = \dfrac{x}{\sqrt{4\eta t}}$ and $\mathrm{erf}(\xi)$ is the error function given by

$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} \mathrm{e}^{-z^2} \mathrm{d}z \tag{4.4}$$

Having known $B(2\sqrt{4\eta t}, t) \approx 0.995 B_0$ and $\int_{-2}^{2} \left[1 - \mathrm{erf}^2(\xi)\right] \mathrm{d}\xi \approx 1.592$, estimate the dissipation rate of the magnetic energy.