Basic Space Plasmas Physics
Assignment 5

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(Solve three out of the four problems! If time permits, you can solve all.)

1 Sound Waves

Before the discussion of MHD waves, let us review the theory of sound waves in ordinary air. Neglecting viscosity, we can write the Navier-Stokes equation as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.1}$$

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right] = -\nabla p \tag{1.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(p\rho^{-\gamma}) = 0 \tag{1.3}$$

The air is in stationary equilibrium state with uniform ρ_0 and p_0. To discuss the sound wave, we take a wave dependence of the form

$$\exp\left(-\mathrm{i}\omega t + \mathrm{i}\vec{k} \cdot \vec{r}\right) \tag{1.4}$$

Derive the eigenmatrix, show the dispersion relation, and calculate the phase velocity and group velocity of the sound waves.

2 Ideal MHD Waves

The ideal MHD equations are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{2.1}$$

$$\rho \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla p + \vec{J} \times \vec{B} \tag{2.2}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}} \left(\frac{p}{\rho^{\gamma}} \right) = 0 \tag{2.4}$$

$$\vec{J} = \frac{1}{\mu_0}(\nabla \times \vec{B}) \tag{2.5}$$

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where, $\vec{B}$ is the magnetic field, $\vec{v}$ is the bulk plasma velocity, $\vec{J}$ is the current density, ρ is the mass density, and p is the plasma pressure. We consider the special case of an infinite, uniform medium with $\vec{v}_0 = 0$, $\vec{B}_0 = (0, 0, B_0)$, and $\vec{J} = 0$. Use the small displacements of the form

$$(\hat{\rho}, \hat{v}_x, \hat{v}_y, \hat{v}_z, \hat{B}_x, \hat{B}_y, \hat{B}_z, \hat{p}) \quad \exp\left(-\mathrm{i}\omega t + \mathrm{i}\vec{k} \cdot \vec{r}\right) \tag{2.6}$$

where $\hat{\rho}, \hat{v_x}, \hat{v_y}, \hat{v_z}, \hat{B_x}, \hat{B_y}, \hat{B_z}, \hat{p}$ are small displacements' magnitude.

1. Derive the eigenmatrix.
2. Show that the dispersion relation is given by

$$\omega^2 \left[\omega^2 - (\vec{k} \cdot \vec{v}_A)^2\right] \left\{\omega^4 - \omega^2 k^2 (v_A^2 + c_s^2) + k^2 c_s^2 (\vec{k} \cdot \vec{v}_A)^2\right\} = 0 \tag{2.7}$$

where $\vec{k}$ is the wave vector, $\vec{v}_A = \sqrt{\dfrac{B_0^2}{\mu_0 \rho_0}} \dfrac{\vec{B}_0}{B_0}$, and $c_s^2 = \dfrac{\gamma p_0}{\rho_0}$.

Hint: An example of the eigenmatrix is given by

$$\begin{pmatrix}
-\mathrm{i}\omega & \mathrm{i}\rho_0 k_x & \mathrm{i}\rho_0 k_y & \mathrm{i}\rho_0 k_z & 0 & 0 & 0 & 0 \\
0 & -\mathrm{i}\omega\rho_0 & 0 & 0 & -\mathrm{i}k_z B_0/\mu_0 & 0 & \mathrm{i}k_x B_0/\mu_0 & \mathrm{i}k_x \\
0 & 0 & -\mathrm{i}\omega\rho_0 & 0 & 0 & -\mathrm{i}k_z B_0/\mu_0 & \mathrm{i}k_y B_0/\mu_0 & \mathrm{i}k_y \\
0 & 0 & 0 & -\mathrm{i}\omega\rho_0 & 0 & 0 & 0 & \mathrm{i}k_z \\
0 & -\mathrm{i}k_z B_0 & 0 & 0 & -\mathrm{i}\omega & 0 & 0 & 0 \\
0 & 0 & -\mathrm{i}k_z B_0 & 0 & 0 & -\mathrm{i}\omega & 0 & 0 \\
0 & \mathrm{i}k_x B_0 & \mathrm{i}k_y B_0 & 0 & 0 & 0 & -\mathrm{i}\omega & 0 \\
\mathrm{i}\omega\gamma p_0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\omega\rho_0
\end{pmatrix} \tag{2.8}$$

3 MHD Shocks

Derive the following expression for the ration of downstream to upstream tangential magnetic field component through a MHD discontinuity

$$\frac{(B_t)_2}{(B_t)_1} = r \frac{(v_n^2)_1 - (c_{int}^2)_1}{(v_n^2)_1 - r(c_{int}^2)_1} \tag{3.1}$$

where $r = (v_n)_1/(v_n)_2 = \rho_2/\rho_1$ is the compression ratio and $c_{int} = (B_n)_1/(\rho_1\mu_0)^{1/2}$ the upstream intermediate speed. Use the derivation of the tangential momentum jump condition and the condition that the tangential electric field is constant through the shock.

4 de Hoffmann-Teller Frame

The upstream de Hoffmann-Teller velocity is given by

$$\vec{V} = -\frac{\vec{n} \times (\vec{B} \times \vec{V}_{in})}{\vec{n} \cdot \vec{B}} \tag{4.1}$$

Show that this is also the de Hoffmann-Teller velocity in the region downstream, *i.e.*, that automatically the downstream flow is field aligned when transforming into the upstream de Hoffmann-Teller frame.