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DECAY OF VORTEX VELOCITY AND DIFFUSION OF TEMPERATURE IN A GENERALIZED SECOND GRADE FLUID *

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Abstract: *The fractional calculus approach in the constitutive relationship model of viscoelastic fluid was introduced. The velocity and temperature fields of the vortex flow of a generalized second fluid with fractional derivative model were described by fractional partial differential equations. Exact analytical solutions of these differential equations were obtained by using the discrete Laplace transform of the sequential fractional derivatives and generalized Mittag-Leffler function. The influence of fractional coefficient on the decay of vortex velocity and diffusion of temperature was also analyzed.*

Key words: generalized second grade fluid; fractional calculus; unsteady flow; temperature field; generalized Mittag-Leffler function

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Introduction

Viscoelastic flows are prevalent in stiring, mixing and chemical reaction of dilute polymer solutions, which often go with heat transfer. It is of very important significance to study the

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mechanism of viscoelastic fluid and heat flow in many industry fields, such as oil exploitation, chemical and food industry and bio-engineering[1]. Supposing linear constitutive relationship, Fetecau *et al*.[2] studied the vortex velocity field and temperature field in second grade fluid. In their report, the constitutive relationship employed has the following form:

$$\tau(t) = \mu\varepsilon(t) + E\frac{\partial\varepsilon(t)}{\partial t}, \tag{1}$$

where τ is the stress, ε is the strain, μ is the coefficient of viscosity, E is the viscoelastic coefficient.

Recently fractional calculus has encountered much success in the description of complex dynamics, such as relaxation, oscillation, diffusion, wave and viscoelastic behavior. Bagley[3], Friedrich[4], Huang Jun-qi[5], He Guang-yu[6], Xu Ming-yu[7,8] and Tan Wen-chang[9~12] *et al*. separately used fractional calculus to handle various rheology problems, and made great achievements. Jiang Ti-qian *et al*. used fractional calculus to analyze the experimental data of viscoelastic colloid, and got good result[13,14].

Generally the constitutive relationship of viscoelastic second order fluids has the form as follows:

$$\tau(t) = \mu\varepsilon(t) + \alpha_1 D_t^{\beta}[\varepsilon(t)], \tag{2}$$

where α_1 is the viscoelastic parameter, β is the fractional coefficient and D_t^{β} is the Riemann-Liouvill fractional calculus operator and may be defined as[15]

$$D_t^{\beta}[f(t)] = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\frac{f(\tau)}{(t-\tau)^{\beta}}\mathrm{d}\tau \qquad (0 < \beta < 1), \tag{3}$$

where $\Gamma(\cdot)$ is Gamma function. While $\beta = 1$, Eq.(2) may be simplified as Eq.(1), which is the classical linear model of second grade fluid; and while $\alpha_1 = 0$ or $\beta = 0$, the constitutive relationship describes complete viscous Newtonian fluid.

In this paper we will study the vortex velocity field and temperature field in a generalized second order fluid by using the constitutive relationship given by Eq.(2). Exact analytic solutions of these differential equations are obtained by using Hankel integral transform, inverse Laplace transform skill and generalized Mittag-Leffler function. Moreover, we successfully analyze how the fractional coefficent β influences the velocity field and temperature field. Many classical results can be special examples of our results, such as the decay of vortex velocity and propagation of a heat wave in a second grade fluid studied by Fetecau[2], and vortex velocity field in classical viscous Newtonian fluid. This provides a new analytical tool for further study of viscoelastic fluid and heat flow.

1 Mathematical Model

We consider a circular motion of generalized second grade fluid whose velocity field, in a system of cylindrical coordinates (r, θ, z), is of the form $v_r = 0, v_\theta = \omega(r,t), v_z = 0$. Here we assume that the initial distribution of the velocity is that of a potential vortex of circulation Γ_0 and the flow is symmetrical to axis.

On the basis of above analysis the constitutive relationship of the generalized second grade fluid for this flow is

$$\tau_{r\theta} = \mu r\frac{\partial}{\partial r}\Big[\frac{\omega(r,t)}{r}\Big] + \alpha_1 D_t^{\beta}\Big[r\frac{\partial}{\partial r}\Big(\frac{\omega(r,t)}{r}\Big)\Big]. \tag{4}$$

Momentum equation is

$$\rho \frac{\partial \omega(r,t)}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\theta}), \tag{5}$$

where $\tau(r,\theta)$ is the component of stress, ρ,μ separately denote density and coefficient of viscosity of the fluid. Substituting Eq.(4) into Eq.(5), we obtain

$$\frac{\partial \omega(r,t)}{\partial t} = (\nu + \alpha D_t^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{6}$$

where $\nu = \mu/\rho, \alpha = \alpha_1/\rho$. The initial condition is

$$\omega(r,0) = \Gamma_0/(2\pi r) \tag{7}$$

and the natural conditions are

$$\omega(r,t),\ \frac{\partial \omega(r,t)}{\partial r} \to 0 \qquad \text{as } r \to \infty, t > 0. \tag{8}$$

Further, we consider that there exists temperature field during vortex flow in a generalized second grade fluid. Its initial distribution and natural conditions are similarly assumed to be of the same forms with Ref.[2], i.e.,

$$\theta(r,0) = \theta_0/(2\pi r), \tag{9}$$

$$\theta(r,t),\ \frac{\partial \theta(r,t)}{\partial r} \to 0 \quad \text{as} \quad r \to \infty, t > 0. \tag{10}$$

The energy equation, when the Fourier's law of heat conduction is considered, may be written in the form[2]

$$\frac{\partial \theta(r,t)}{\partial t} = \beta_1\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\theta(r,t) + \frac{\nu}{c}\left[\frac{\partial \omega(r,t)}{\partial r} - \frac{\omega(r,t)}{r}\right]^2 + \frac{h(r,t)}{c}, \tag{11}$$

where $h(r,t)$ is the radiant heating which is neglected in this paper, c is the specific heat and $\beta_1 = k/(\rho c)$, where k is the conductivity, which is assumed to be constant. Eq.(11) is formally identical to the energy equation for a Newtonian fluid and classical second grade fluid. However, the temperature field given by Eq.(11) is different from the Newtonian case and the classical second grade case because the velocity distribution $\omega(r,t)$ is different.

2 Velocity Field

Let us introduce dimensionless variables $\omega^* = \omega\nu/\Gamma_0^2$, $r^* = r\Gamma_0/\nu$, $t^* = t\Gamma_0^2/\nu$. Using Eq.(3) and the first mean value theorem of the integral, it can be easily proved[7,9,16] that the operator D_t^{β} has a fractional time dimensions $[\nu/\Gamma_0^2]^{-\beta}$. Thus, the dimensionless equation is obtained as follows (for brevity the dimensionless mark "$*$" are omitted here):

$$\frac{\partial \omega(r,t)}{\partial t} = (1 + \eta D_t^{\beta})\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{12}$$

$$\omega(r,t),\ \frac{\partial \omega(r,t)}{\partial r} \to 0 \qquad \text{as } r \to \infty, t > 0, \tag{13}$$

$$\omega(r,0) = 1/(2\pi r), \tag{14}$$

where $\eta = \alpha \frac{\Gamma_0^2}{\nu^2}$. Considering the fractional order Eq.(12) has the integral order initial condition (14), we define the fractional calculus operator D_t^{β} to be of the form as Sequential Fractional Derivatives[15]. In order to get the exact solution to those equations, we introduce the Hankel

transform as follows[17]:
Hankel transform

$$\omega_{\mathrm{h}}(\xi,t) = \int_0^{\infty} r\mathrm{J}_1(\xi r)\omega(r,t)\mathrm{d}r; \tag{15}$$

Inverse Hankel transform

$$\omega(r,t) = \int_0^{\infty} \xi\mathrm{J}_1(\xi r)\omega_{\mathrm{h}}(\xi,t)\mathrm{d}\xi, \tag{16}$$

where $\mathrm{J}_1(\xi r)$ is the first kind Bessel function of the first order.

Applying Hankel transform principle to Eqs.(12) and (14), we can obtain

$$\frac{\mathrm{d}\omega_{\mathrm{h}}(\xi,t)}{\mathrm{d}t} + (1+\eta D_t^{\beta})[\xi^2\omega_{\mathrm{h}}(\xi,t)] = 0, \tag{17}$$

$$\omega_{\mathrm{h}}(\xi,0) = 1/(2\pi\xi). \tag{18}$$

Letting $\overline{\omega_{\mathrm{h}}}(\xi,s) = L\{\omega_{\mathrm{h}}(\xi,t)\} = \int_0^{\infty}\mathrm{e}^{-st}\omega_{\mathrm{h}}(\xi,t)\mathrm{d}t$ be the image function of $\omega_{\mathrm{h}}(\xi,t)$, where s is a transform parameter and using Laplace transform principle of Sequential Fractional Derivatives to Eq.(17)[15], we can obtain

$$\overline{\omega_k}(\xi,s) = \frac{1}{2\pi\xi}(1+\eta\,\xi^2 s^{\beta-1})\frac{1}{s+\eta\,\xi^2 s^{\beta}+\xi^2}. \tag{19}$$

In order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method to get $\overline{\omega_k}(\xi,s)$. First, we rewrite Eq.(19) as a series form

$$\begin{aligned}\overline{\omega_k}(\xi,s) &= \frac{1}{2\pi\xi}(1+\eta\,\xi^2 s^{\beta-1})\frac{1}{\xi^2}\sum_{k=0}^{\infty}(-1)^k\xi^{2(k+1)}\frac{s^{-\beta k-\beta}}{(s^{1-\beta}+\eta\,\xi^2)^{k+1}} = \\ &\frac{1}{2\pi\xi^3}\sum_{k=0}^{\infty}(-1)^k\xi^{2(k+1)}\frac{s^{-\beta k-\beta}}{(s^{1-\beta}+\eta\,\xi^2)^{k+1}} + \\ &\frac{\eta}{2\pi\xi}\sum_{k=0}^{\infty}(-1)^k\xi^{2(k+1)}\frac{s^{-\beta k-1}}{(s^{1-\beta}+\eta\,\xi^2)^{k+1}}.\end{aligned} \tag{20}$$

Applying the inverse Laplace transform term by term with Eq.(20), we obtain

$$\begin{aligned}\omega_{\mathrm{h}}(\xi,t) = \frac{1}{2\pi\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\xi^{2k}t^k[&E_{1-\beta,1+\beta k}^{(k)}(-\eta\,\xi^2 t^{1-\beta}) + \\ &\eta t^{1-\beta}E_{1-\beta,2+\beta(k-1)}^{(k)}(-\eta\,\xi^2 t^{1-\beta})],\end{aligned} \tag{21}$$

in which $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$ denotes generalized Mittag-Leffler function[15]. To obtain Eq.(21), we used the following property of the generalized Mittag-Leffler function's inverse Laplace transform:

$$L^{-1}\left\{\frac{n!\,s^{\lambda-\mu}}{(s^{\lambda}\mu c)^{n+1}}\right\} = t^{\lambda n+\mu-1}E_{\lambda,\mu}^{(n)}(\pm ct^{\lambda}) \qquad (\mathrm{Re}(s) > |c|^{1/\lambda}). \tag{22}$$

Applying inverse Hankel transform to Eq.(21), we obtain the exact solution to the velocity field:

$$\begin{aligned}\omega(r,t) = \frac{1}{2\pi}\int_0^{\infty}\mathrm{J}_1(r\xi)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\xi^{2k}t^k[&E_{1-\beta,1+\beta k}^{(k)}(-\eta\,\xi^2 t^{1-\beta}) + \\ &\eta\,\xi^2 t^{1-\beta}E_{1-\beta,2+\beta(k-1)}^{(k)}(-\eta\,\xi^2 t^{1-\beta})]\mathrm{d}\xi.\end{aligned} \tag{23}$$

Especially, when $\eta = 0$, Eq.(19) can be simplified as

$$\overline{\omega_k}(\xi, s) = \frac{1}{2\pi\xi(s + \xi^2)}. \tag{24}$$

Separately applying inverse Laplace transform and inverse Hankel transform to s and ξ in Eq.(24), we obtain

$$\omega(r, t) = \frac{1}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4t}\right)\right] \tag{25}$$

which is the Reiner-Rivlin dimensionless solution of the classical viscous Newtonian fluid problem[2].

If we set $\beta = 1$, Eq.(19) can be simplified as

$$\overline{\omega_k}(\xi, s) = \frac{1}{2\pi\xi}\frac{1 + \eta\xi^2}{s + \eta\xi^2 s^\beta + \xi^2}. \tag{26}$$

Separately applying inverse Laplace transform and inverse Hankel transform to s and ξ in Eq.(26), we obtain

$$\omega(r, t) = \frac{1}{2\pi}\int_0^\infty \mathrm{J}_1(r\xi)\exp\left(-\frac{\xi^2}{1 + \eta\xi^2}t\right)\mathrm{d}\xi \tag{27}$$

which is the dimensionless velocity solution of a second grade fluid obtained by Fetecau[2].

3 Temperature Field

Let us introduce dimensionless variables: $\theta^* = \frac{\theta\nu}{\theta_0\Gamma_0}$, $\omega^* = \frac{\omega\nu}{\Gamma_0^2}$, $r^* = \frac{r\Gamma_0}{\nu}$, $t^* = \frac{t\Gamma_0^2}{\nu}$, and then Eqs.(9) ~ (11) can be changed into dimensionless equations as follows (for brevity the dimensionless mark " * " are omitted here):

$$\frac{\partial\theta(r, t)}{\partial t} = \beta_2\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\theta(r, t) + \eta_1\left[\frac{\partial\omega(r, t)}{\partial r} - \frac{\omega(r, t)}{r}\right]^2, \tag{28}$$

$$\theta(r, t), \frac{\partial\theta(r, t)}{\partial r} \to 0 \qquad \text{as } r \to \infty, t > 0, \tag{29}$$

$$\theta(r, 0) = 1/(2\pi r), \tag{30}$$

where $\beta_2 = \frac{\beta_1}{\nu}$, $\eta_1 = \frac{\Gamma_0^3}{c\nu\theta_0}$. Letting

$$f(r, t) = \eta_1\left[\frac{\partial\omega(r, t)}{\partial r} - \frac{\omega(r, t)}{r}\right]^2 \tag{31}$$

and Eq.(28) can be changed into

$$\frac{\partial\theta(r, t)}{\partial t} = \beta_2\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}\right)\theta(r, t) + f(r, t). \tag{32}$$

In order to get the exact solution to those equations, we make the Hankel transform to r in Eqs.(30) and (32). According to the format of Eq.(32), we use the Hankel transform as follows:

Hankel transform

$$\theta_{\mathrm{h}}(\xi, t) = \int_0^\infty r\mathrm{J}_0(\xi r)\theta(r, t)\mathrm{d}r; \tag{33}$$

Inverse Hankel transform

$$\theta(r, t) = \int_0^\infty \xi\mathrm{J}_0(\xi r)\theta_{\mathrm{h}}(\xi, t)\mathrm{d}\xi, \tag{34}$$

where $J_0(\xi r)$ is the first kind Bassel function of the zero order.

Applying Hankel transform to Eq. (32) and making use of natural conditions Eq. (29), we obtain

$$\frac{d\theta_h(\xi,t)}{dt} + \beta_2\xi^2\theta_h(\xi,t) = f_h(\xi,t), \tag{35}$$

where $f_h(\xi,t)$ is the Hankel transform image function of $f(r,t)$. Substituting Eq. (23) into Eq. (31), we obtain

$$\begin{aligned} f_h(\xi,t) = {} & \frac{\eta_1}{4\pi^2}\int_0^\infty rJ_0(\xi r)\times \\ & \left\{\int_0^\infty xJ_2(xr)\sum_{k=0}^\infty \frac{(-1)^k}{k!}x^{2k}t^k[E_{1-\beta,1+\beta k}^{(k)}(-\eta x^2t^{1-\beta}) + \right. \\ & \left.\eta x^2t^{1-\beta}E_{1-\beta,2+\beta(k-1)}^{(k)}(-\eta x^2t^{1-\beta})]dx\right\}^2 dr. \end{aligned} \tag{36}$$

To obtain Eq. (36), we used the following property of Bessel function[17]:

$$xrJ_1'(xr) = J_1 - xrJ_2(xr).$$

Applying Hankel transform to the initial condition (30), we can obtain

$$\theta_h(\xi,0) = 1/(2\pi\xi). \tag{37}$$

Combining Eq. (35) with Eq. (37), we can easily get

$$\theta_h(\xi,t) = e^{-\beta_2\xi^2 t}\left[\frac{1}{2\pi\xi} + \int_0^t f_h(\xi,\tau)e^{\beta_2\xi^2\tau}d\tau\right]. \tag{38}$$

Then, making inverse Hankel transform to the equation above, we obtain

$$\begin{aligned} \theta(r,t) = {} & \frac{1}{2\pi}\int_0^\infty e^{-\beta_2\xi^2 t}J_0(\xi r)d\xi + \\ & \int_0^\infty \xi J_0(\xi r)\cdot\left[\int_0^t f_h(\xi,\tau)e^{-\beta_2\xi^2(t-\tau)}d\tau\right]d\xi. \end{aligned} \tag{39}$$

Substituting Eq. (36) into Eq. (39), we can obtain the temperature field as follows:

$$\begin{aligned} \theta(r,t) = {} & \frac{1}{2\pi}\int_0^\infty e^{-\beta_2\xi^2 t}J_0(\xi r)d\xi + \\ & \frac{\eta_1}{4\pi^2}\int_0^\infty \xi J_0(r\xi)\int_0^t e^{-\beta_2\xi^2(t-\tau)}\int_0^\infty rJ_0(\xi r)g^2(r,\tau)dr d\tau d\xi, \end{aligned} \tag{40}$$

where

$$\begin{aligned} g(r,\tau) = {} & \int_0^\infty xJ_2(rx)\sum_{k=0}^\infty \frac{(-1)^k}{k!}x^{2k}\tau^k[E_{1-\beta,1+\beta k}^{(k)}(-\eta x^2t^{1-\beta}) + \\ & \eta x^2t^{1-\beta}E_{1-\beta,2+\beta(k-1)}^{(k)}(-\eta x^2t^{1-\beta})]dx. \end{aligned} \tag{41}$$

4 Discussion and Conclusions

We obtained the analytical solutions of the vortex velocity and temperature fields in a generalized second grade fluid above through Eqs. (23) and (40). Using the Matlab software, we simulate the influence of the fractional coefficient β on the velocity and temperature distribution. Figs. 1 and 2 show that the vortex velocity spacial distributions for the different values of time and fractional coefficient. It can be seen that the velocity increases with r and gets to the maximum value at some position because there exists a potential vortex of circulation Γ_0, and then it begins to decay until to zero since the viscidity of the fluid makes the influence of vortex on velocity become more and more weak. In general the variety trend of the velocity curves are similar with that in the classical second grade fluid described in Ref.[2]. Fig.1 shows the velocity distribution at several selected β. We can see that when β becomes larger, the curve gets acuter, that is, the maximum value of the velocity becomes larger, the position r corresponding to the maximum velocity becomes smaller, and subsequently the vortex velocity decays more quickly. It can be concluded that the larger the β is, the more viscoelastic the fluid is, and the more similar to the classical second grade fluid. Especially, when $\beta = 1$, it completely changes into the classical second grade fluid. On the other hand, the smaller the β is, the more viscous the fluid is, and if $\beta = 0$, the fluid can be simplify as the viscous Newtonian fluid. Fig.2 shows the velocity distribution at several selected t. It can be found that the vortex velocity at the same location becomes smaller with the time increases because of the viscosity of the fluid.

Figures 3 and 4 show the distributions of the temperature field $\theta(r, t)$ at several selected parameters. Fig.3 is the temperature distribution at several selected β. It can be found that the

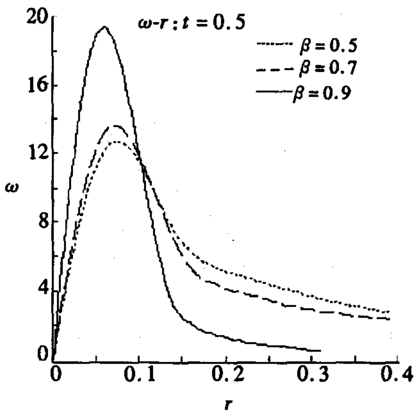

Fig.1 Vortex velocity distribution with different β

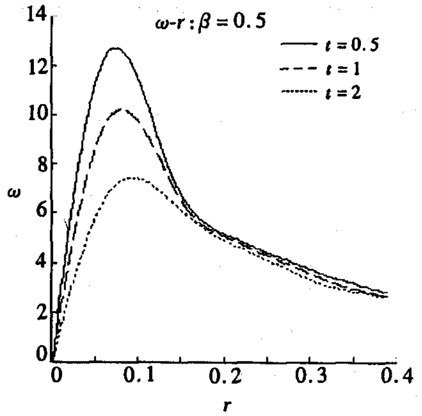

Fig.2 Vortex velocity distribution with different t

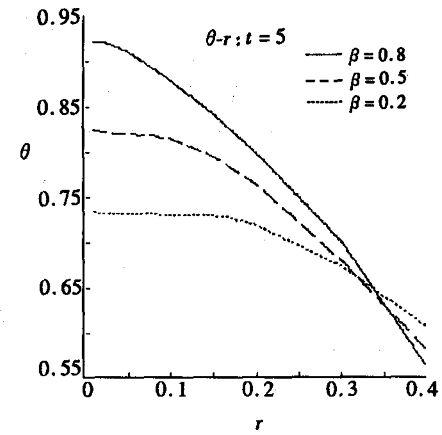

Fig.3 Temperature distribution with different β

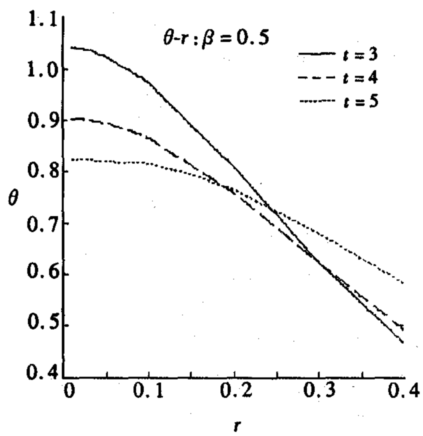

Fig.4 Temperature distribution with different t

larger β is, the more quickly the temperature decays with r, and the more viscoelastic the fluid is. On the other hand, the smaller the β is, the more slowly the temperature decays, and the more viscous the fluid is. Fig.4 shows the temperature distribution at several selected time, from which we can see the temperature field decays more and more slowly with time, that is because of the viscosity of the fluid.

We can also find from the four figures that when t and r become large enough, no matter the velocity field or the temperature field will decay until reaching zero, furthermore, the larger the β is, the more quickly the fields decay. So it means that the viscoelastic property of the fluid goes against the diffusions of the velocity, the temperature, the vortex and the heat wave decay in time and space.

In this paper, the problems on the decay of vortices in the viscous Newtonian fluid or the second grade fluid were expanded to the generalized second grade fluid with fractional derivatives and the exact solutions of the decay of the vortex velocity and the diffusion of the temperature were also successfully obtained. The model and the analytical method employed in this paper have been shown to be useful for the theory analyses of viscoelasitic fluid.

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