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The Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivative model

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Abstract

The Rayleigh–Stokes problem for a generalized second grade fluid subject to a flow on a heated flat plate and within a heated edge was investigated. For description of such a viscoelastic fluid, fractional calculus approach in the constitutive relationship model was used. Exact solutions of the velocity and temperature fields were obtained using the Fourier sine transform and the fractional Laplace transform. The well-known solutions of the Stokes' first problem for a viscous Newtonian fluid, as well as those corresponding to a second grade fluid, appear in limiting cases of our results.

Keywords: Generalized second grade fluid; Fractional calculus; Rayleigh–Stokes problem; Heat transfer

1. Introduction

There are very few cases in which the exact solutions of Navier–Stokes equations can be obtained. These are even rare if the constitutive relations for viscoelastic fluids are considered. However, the interest in viscoelastic flows has grown considerably, due largely to the demands of such diverse areas as biorheology, geophysics, chemical and petroleum industries [26]. Because of the difficulty to suggest a single model, which exhibits all properties of viscoelastic fluids, they cannot be described as simply as Newtonian fluids. For this reason many models of constitutive equations have been proposed. Recently fractional calculus has encountered much success in the description of viscoelasticity. The starting point of the fractional derivative model of viscoelastic fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann–Liouville fractional calculus operator. This generalization allows one to define precisely non-integer order integrals or derivatives. Bagley [1], Friedrich [5], Huang Junqi [7], He Guangyu [6], Xu [24,25] and Tan [19–21,23] have sequentially introduced the fractional calculus approach into various rheology problems. Fractional derivatives have been found to be quite flexible in describing viscoelastic behaviors.

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The first problem of Stokes for the flat plate as well as the Rayleigh–Stokes problem for an edge has received much attention because of its practical importance [17,18,27]. This unsteady flow problem examines the diffusion of vorticity in a half-space filled with a viscous incompressible fluid that is set to motion when an infinite flat plate suddenly assumes a constant velocity parallel to itself from rest. By means of the similarity by transformation of variables, the exact solution corresponding to a Newtonian fluid was obtained in an elegant form by Stokes. But for a second grade fluid, a strict similarity solution does not exist [16]. Further, the equation of motion for such a fluid is a higher order than the Navier–Stokes equation and thus, in general one needs conditions in addition to the usual adherence boundary condition. Rajagopal firstly investigated this problems and gave a few exact solutions [2,10–14].

However, the determination of the temperature distribution within a fluid when the internal friction is not negligible is of utmost importance. The thermal convection of a second grade fluid subject to some unidirectional flows was studied by Bandelli [2]. Recently, Fetecau extended the Rayleigh–Stokes problem to that for a heated second grade fluids [4]. In this paper, the Rayleigh–Stokes problem for a heated generalized second grade fluid was investigated. The temperature distribution in a generalized second grade fluid subject to a flow on a heated flat plate and within a heated edge was determined using the Fourier sine transform and fractional Laplace transformuential fractional derivatives. Some classical and previous results can be regarded as particular cases of our results, such as the classical solution of the first problem of Stokes for Newtonian viscous fluid and that for a heated second grade fluid.

2. Constitutive equations

For a second grade fluid, the extra stress tensor $\mathbf{T}$ is given by the constitutive equation [15]:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the hydrostatic pressure, $\mathbf{I}$ the identity tensor. α_1 and α_2 are normal stress moduli. $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical tensors defined through

$$\mathbf{A}_1 = grad\,\mathbf{V} + (grad\,\mathbf{V})^{\mathrm{T}}, \tag{2}$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(grad\,\mathbf{V}) + (grad\,\mathbf{V})^{\mathrm{T}}\mathbf{A}_1, \tag{3}$$

where $\mathrm{d}/\mathrm{d}t$ denotes the material time derivative, $\mathbf{V}$ is the velocity and *grad* the gradient operator.

The second grade fluid given by Eq. (1) is compatible with thermodynamics, then the material moduli must meet the following restrictions [3]:

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{4}$$

For a generalized second grade fluids, Eq. (1) is still fit, but $\mathbf{A}_2$ should be defined as follows [1,7,24,22]:

$$\mathbf{A}_2 = D_t^{\beta}\mathbf{A}_1 + \mathbf{A}_1(grad\,\mathbf{V}) + (grad\,\mathbf{V})^{\mathrm{T}}\mathbf{A}_1, \tag{5}$$

where β is the fractional coefficient and D_t^{β} is the Riemann–Liouvill fractional calculus operator and may be defined as [9]:

$$D_t^{\beta}[f(t)] = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \frac{f(\tau)}{(t-\tau)^{\beta}}\,\mathrm{d}\tau \quad (0 < \beta < 1) \tag{6}$$

where $\Gamma(\bullet)$ is gamma function. While $\beta = 1$, Eq. (5) can be simplified as Eq. (3), and while $\alpha_1 = 0$ and $\beta = 0$, the constitutive relationship describes complete viscous Newtonian fluid.

The equation of motion in the absence of body forces can be described as:

$$\rho\frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = \nabla\cdot\mathbf{T}, \tag{7}$$

where ρ is the density of fluid and $\mathrm{D}/\mathrm{D}t$ is the material derivative. The fluid being incompressible it can undergo only isochoric motion and hence

$$\nabla\cdot\mathbf{V} = 0. \tag{8}$$

3. Stokes' first problem for a heated flat plate

Suppose that a generalized second grade fluid, at rest, occupies the space above an infinitely extended plate in the (y, z)—plane. At time $t = 0^+$ the plane suddenly moves in its plane with a constant velocity U. Let T_0 denotes temperature of the plate for $t \geqslant 0$, and suppose the temperature of the fluid at the moment $t = 0$ is zero. By the influence of shear and of heat conduction the fluid, above the plate, is gradually moved and heated. The velocity field will be of the form

$$\mathbf{V} = u(x, t)\mathbf{j}, \tag{9}$$

where u is the velocity in the y coordinate direction and $\mathbf{j}$ denotes a unit vector along the y-coordinate direction. And the temperature field is of the form $\theta = \theta(x, t)$.

Substituting Eq. (9) into Eqs. (1), (2) and (5), we obtain the stress component

$$T_{xy} = T_{yx} = \mu \frac{\partial u(x,t)}{\partial x} + \alpha_1 D_t^{\beta} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{10}$$

and other stress components are zero. Inserting Eqs. (9) and (10) into (7) yields

$$\frac{\partial u(x,t)}{\partial t} = (v + \alpha D_t^{\beta}) \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{11}$$

where $v = \mu/\rho$, $\alpha = \alpha_1/\rho$.

The energy equation, when the Fourier's law of heat conduction is considered, may be written in the form as follows:

$$\frac{k}{c\rho} \frac{\partial^2 \theta(x,t)}{\partial x^2} + \frac{v}{c}\left[\frac{\partial u(x,t)}{\partial x}\right]^2 + \frac{r(x,t)}{\rho c} = \frac{\partial \theta(x,t)}{\partial t}, \tag{12}$$

where $r(x, t)$ is the radiant heating, which is neglected in this paper, c is the specific heat and k is the conductivity, which is assumed to be constant.

The corresponding initial and boundary conditions are

$$u(x, 0) = 0 \quad \text{for } x > 0, \tag{13}$$

$$u(0, t) = U \quad \text{for } t > 0, \tag{14}$$

respectively,

$$\theta(x, 0) = 0 \quad \text{for } x > 0, \tag{15}$$

$$\theta(0, t) = T_0 \quad \text{for } t \geqslant 0. \tag{16}$$

Moreover, the natural conditions

$$u(x,t), \frac{\partial u(x,t)}{\partial x} \to 0 \quad \text{for } x \to 0 \tag{17}$$

and

$$\theta(x,t), \frac{\partial \theta(x,t)}{\partial x} \to 0 \quad \text{for } x \to 0 \tag{18}$$

also have to be satisfied [4].

3.1. Solution of velocity field

Employing the non-dimensional quantities

$$u^* = \frac{u}{U}, \quad x^* = \frac{xU}{v}, \quad t^* = \frac{tU^2}{v}, \quad \eta = \frac{\alpha U^2}{v^2}, \tag{19}$$

in which U and v/U^2 denote characteristic velocity and time, we obtain the dimensionless motion equation as follows (for brevity the dimensionless mark "*" are omitted here)

$$\frac{\partial u(x,t)}{\partial t} = (1+\eta D_t^{\beta})\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{20}$$

$$u(x,0)=0 \quad \text{for } x>0, \tag{21}$$

$$u(0,t)=1 \quad \text{for } t>0, \tag{22}$$

$$u(x,t), \frac{\partial u(x,t)}{\partial x} \to 0 \quad \text{for } x \to 0. \tag{23}$$

If we multiply both sides of Eqs. (20) and (21) by $\sqrt{2/\pi}\sin(x\xi)$, then integrate with respect to x from 0 to ∞ and use the boundary conditions of Eqs. (22) and (23), we get

$$\frac{\mathrm{d}U(\xi,t)}{\mathrm{d}t} = -\xi^2(1+\eta D_t^{\beta})U(\xi,t) + \sqrt{\frac{2}{\pi}}\,\xi, \tag{24}$$

$$U(\xi,0)=0, \tag{25}$$

where $U(\xi,t)$ denotes the Fourier sine transform of $u(x,t)$ with respect to x. In order to obtain an exact solution of Eq. (24) subject to the initial condition of Eq. (25), the fractional Laplace transform is used. $\overline{U}(\xi,s)=L\{U(\xi,t)\}=\int_0^\infty \mathrm{e}^{-st}U(\xi,t)\,\mathrm{d}t$ is the image function of $U(\xi,t)$, where s is a transform parameter. Using Laplace transform principle of Sequential Fractional Derivatives [9,8], we have

$$\overline{U}(\xi,s) = \sqrt{\frac{2}{\pi}}\frac{\xi}{s(s+\eta\xi^2 s^{\beta}+\xi^2)}. \tag{26}$$

In order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method here. Firstly, we rewrite Eq. (26) as a series form

$$\overline{U}(\xi,s) = \sqrt{\frac{2}{\pi}}\frac{1}{\xi}\sum_{k=0}^{\infty}(-1)^k\xi^{2(k+1)}\frac{s^{-\beta k-\beta-1}}{(s^{1-\beta}+\eta\xi^2)^{k+1}}. \tag{27}$$

Then, applying the inversion formulae term by term for the Laplace transform, Eq. (27) yields

$$U(\xi,t) = \sqrt{\frac{2}{\pi}}\frac{1}{\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\xi^{2(k+1)}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}(-\eta\xi^2 t^{1-\beta}) \tag{28}$$

in which $E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}t^k/\Gamma(\alpha k+\beta)$ denotes generalized Mittag–Leffle function. Here, we used the following property of the generalized Mittag–Leffle function's inverse Laplace transform [24]

$$L^{-1}\left\{\frac{n!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{n+1}}\right\} = t^{\lambda n+\mu-1}E^{(n)}_{\lambda,\mu}(\pm ct^{\lambda}) \quad (\mathrm{Re}(s)>|c|^{1/\lambda}). \tag{29}$$

Inverting Eq. (28) by means of Fourier sine transform, we obtain an exact solution to the velocity field as follows:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\xi x)}{\xi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\xi^{2(k+1)}t^{k+1}E^{(k)}_{1-\beta,2+\beta k}(-\eta\xi^2 t^{1-\beta})\,\mathrm{d}\xi. \tag{30}$$

In a special case, when $\beta=1$, corresponding to the Stokes' first problem for a heated second grade fluid, Eq. (26) can be simplified to

$$\overline{U}(\xi,s) = \sqrt{\frac{2}{\pi}}\frac{\xi}{s(s+\eta\xi^2 s+\xi^2)}. \tag{31}$$

Inversing Eq. (31) by means of fractional Laplace transform and Fourier sine transform, we have

$$u(x,t)=1-\frac{2}{\pi}\int_0^\infty \frac{\sin(\xi x)}{\xi}\exp\left(-\frac{\xi^2}{1+\eta\xi^2}t\right)\mathrm{d}\xi. \tag{32}$$

This is just the dimensionless velocity solution of a second grade fluid obtained by Fetecau [4].

In a special case, when $\beta=0$ and $\alpha=0$, corresponding to the classical version of Stokes' first problem for a Newtonian fluid, Eq. (30) can be simplified to

$$\begin{aligned}u(x,t)&=\frac{2}{\pi}\int_0^\infty \frac{1}{s}(1-\exp(-s^2t)\sin(sx)\,\mathrm{d}s\\&=1-\frac{2}{\pi}\int_0^\infty \frac{1}{s}\exp(-s^2t)\sin(sx)\,\mathrm{d}s.\end{aligned} \tag{33}$$

Eq. (33) can be simplified further as follows [4]:

$$u(x,t)=1-erf\left(\frac{x}{2\sqrt{t}}\right), \tag{34}$$

where $erf(x)$ is the error function of Gauss. This is just the well-known result of Stokes' first problem for a Newtonian fluid.

3.2. Solution of temperature field

Eq. (12) is identical to the energy equation for a Newtonian fluid and classical second grade fluid. However, the temperature field given by Eq. (12) is different from the Newtonian case and the classical second grade case because the velocity distribution $u(x,t)$ is different.

Employing the non-dimensional quantities

$$\theta^*=\frac{\theta}{T_0},\quad v^*=\frac{u}{U},\quad x^*=\frac{xU}{v},\quad t^*=\frac{tU^2}{v},\quad \lambda=\frac{U^2}{cT},\quad Pr=\frac{c\mu}{k}, \tag{35}$$

Eqs. (12), (15), (16) and (18) can reduce to dimensionless equations as follows (for brevity the dimensionless mark "*" are omitted here)

$$\frac{1}{Pr}\frac{\partial^2\theta(x,t)}{\partial x^2}+\lambda\left[\frac{\partial v(x,t)}{\partial x}\right]^2=\frac{\partial\theta(x,t)}{\partial t}, \tag{36}$$

$$\theta(x,0)=0\quad \text{for } x>0, \tag{37}$$

$$\theta(0,t)=1\quad \text{for } t\geqslant 0, \tag{38}$$

$$\theta(x,t),\ \frac{\partial\theta(x,t)}{\partial x}\to 0\quad \text{for } x\to 0. \tag{39}$$

Letting $g(x,t)=\lambda[\partial u(x,t)/\partial x]^2$, Eq. (36) can be rewritten as

$$\frac{1}{Pr}\frac{\partial^2\theta(x,t)}{\partial x^2}+g(x,t)=\frac{\partial\theta(x,t)}{\partial t}, \tag{40}$$

where $g(x,t)$ is a known function because the velocity field $u(x,t)$ is obtained by Eq. (30). Applying Fourier sine transform to Eqs. (37) and (40), we get

$$\frac{\mathrm{d}\theta_s(\xi,t)}{\mathrm{d}t}+\frac{1}{Pr}\xi^2\theta_s(\xi,t)=\sqrt{\frac{2}{\pi}}\frac{1}{Pr}\xi+g_s(\xi,t), \tag{41}$$

$$\theta_s(\xi,0)=0, \tag{42}$$

where $\theta_s(\xi, t)$ and $g_s(\xi, t)$ denote the Fourier sine transforms of $\theta(x, t)$ and $g(x, t)$ with respect to x, respectively. The solution of the ordinary differential equation (41) subject to the initial condition (42) is given by

$$\theta_s(\xi, t) = \mathrm{e}^{-\xi^2 t/Pr} \cdot \int_0^t \left[\sqrt{\frac{2}{\pi}} \frac{1}{Pr} \xi + g_s(\xi, \tau) \right] \mathrm{e}^{\xi^2 \tau/Pr} \, \mathrm{d}\tau. \tag{43}$$

Inverting Eq. (43) by means of Fourier sine transform, we get

$$\theta(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\xi x) \mathrm{e}^{-\xi^2 t/Pr} \cdot \int_0^t \left[\sqrt{\frac{2}{\pi}} \frac{1}{Pr} \xi + g_s(\xi, \tau) \right] \mathrm{e}^{\xi^2 \tau/Pr} \, \mathrm{d}\tau \, \mathrm{d}\xi. \tag{44}$$

In a special case, at rest, it implies that $g_s(\xi, \tau) = 0$. Submitting $g_s(\xi, \tau) = 0$ into Eq. (44) yields

$$\begin{aligned} \theta(x, t) &= \frac{2}{\pi} \int_0^\infty \sin(\xi x) \mathrm{e}^{-\xi^2 t/Pr} \cdot \int_0^t \frac{1}{Pr} \xi \mathrm{e}^{\xi^2 \tau/Pr} \, \mathrm{d}\tau \, \mathrm{d}\xi = 1 - \frac{2}{\pi} \int_0^\infty \frac{1}{\xi} \sin(\xi x) \mathrm{e}^{-\xi^2 t/Pr} \, \mathrm{d}\xi \\ &= 1 - erf\left(\frac{x}{2\sqrt{\frac{t}{Pr}}} \right). \end{aligned} \tag{45}$$

We can find that the temperature distribution in the generalized second fluid is the same as that both for a second grade fluid and a Newtonian one at rest [4].

4. The Rayleigh–Stokes problem for a heated edge

Let us now consider a generalized second grade fluid at rest occupies the space of the first dial of a rectangular edge ($x \geqslant 0, -\infty < y < \infty, z \geqslant 0$). At the moment $t = 0^+$ the extended edge is impulsively brought to the constant velocity U. The two walls of the edge will be again maintained to the temperature T_0. The velocity field will be of the form

$$\mathbf{V} = u(x, z, t)\mathbf{j} \tag{46}$$

and the temperature field $\theta = \theta(x, z, t)$.

Similar to Section 3, the balance of linear momentum and the energy equation reduce to

$$\frac{\partial u(x, z, t)}{\partial t} = (v + \alpha D_t^\beta) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) u(x, z, t) \tag{47}$$

and

$$\frac{\partial \theta(x, z, t)}{\partial t} = \frac{k}{\rho c} \left[\frac{\partial^2 \theta(x, z, t)}{\partial x^2} + \frac{\partial^2 \theta(x, z, t)}{\partial z^2} \right] + \frac{v}{c} f(x, z, t) + \frac{r(x, z, t)}{\rho c}, \tag{48}$$

where $f(x, z, t) = [\partial u(x, z, t)/\partial x]^2 + [\partial u(x, z, t)/\partial z]^2$ is a known function as soon as the velocity field $u(x, z, t)$ is prescribed, $r(x, z, t)$ is the radiant heating, which is neglected in this paper.

The corresponding initial and boundary conditions are

$$u(x, z, 0) = 0 \quad \text{for } x > 0, \quad z > 0, \tag{49}$$

$$u(0, z, t) = u(x, 0, t) = U \quad \text{for } t > 0, \tag{50}$$

respectively,

$$\theta(x, z, 0) = 0 \quad \text{for } x > 0, \quad z > 0, \tag{51}$$

$$\theta(0, z, t) = \theta(x, 0, t) = T_0 \quad \text{for } t > 0. \tag{52}$$

The natural conditions

$$u(x,z,t),\ \frac{\partial u(x,z,t)}{\partial x},\ \frac{\partial u(x,z,t)}{\partial z} \to 0 \quad \text{for } x^2+z^2 \to \infty \tag{53}$$

and

$$\theta(x,z,t),\ \frac{\partial \theta(x,z,t)}{\partial x},\ \frac{\partial \theta(x,z,t)}{\partial z} \to 0 \quad \text{for } x^2+z^2 \to \infty \tag{54}$$

have to be satisfied too.

4.1. Solution of velocity field

Using the non-dimensional quantities given by Eq. (19) and $z^* = zU/v$, Eqs. (47), (49), (50) and (53) reduce to dimensionless equations as follows (for brevity the dimensionless mark "*" are omitted here)

$$\frac{\partial u(x,z,t)}{\partial t} = (1+\eta D_t^{\beta})\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)u(x,z,t), \tag{55}$$

$$u(x,z,0)=0 \quad \text{for } x>0,\ \ z>0, \tag{56}$$

$$u(0,z,t)=u(x,0,t)=1 \quad \text{for } t>0, \tag{57}$$

$$u(x,z,t),\ \frac{\partial u(x,z,t)}{\partial x},\ \frac{\partial u(x,z,t)}{\partial z} \to 0 \quad \text{for } x^2+z^2 \to \infty. \tag{58}$$

Applying the Fourier sine transform and fractional Laplace transform to above equations, we get

$$\overline{U}(\xi,\varsigma,s) = \frac{2(\xi^2+\varsigma^2)}{\pi\xi\varsigma s[s+\eta(\xi^2+\varsigma^2)s^{\beta}+(\xi^2+\varsigma^2)]}, \tag{59}$$

where $\overline{U}(\xi,\varsigma,s)$ denotes the Laplace transform of $U(\xi,\varsigma,t)$ with respect to t and $U(\xi,\varsigma,t)$ the Fourier sine transform of $u(x,z,t)$ with respect to x, z.

Inverting Eq. (59) by means of Laplace transform and Fourier sine transform, we get

$$u(x,z,t) = \frac{4}{\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{\sin(\xi x)\sin(\varsigma z)}{\xi\varsigma}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\xi^2+\varsigma^2)^{k+1}t^{k+1} \times E^{(k)}_{1-\beta,2+\beta k}[-\eta(\xi^2+\varsigma^2)t^{1-\beta}]\,\mathrm{d}\xi\,\mathrm{d}\varsigma. \tag{60}$$

In a special case, when $\beta=1$, Eq. (59) reduces to

$$\overline{U}(\xi,\varsigma,s) = \frac{2(\xi^2+\varsigma^2)}{\pi\xi\varsigma s[s+\eta(\xi^2+\varsigma^2)s+(\xi^2+\varsigma^2)]}. \tag{61}$$

Inverting Eq. (61) by means of the Laplace transform and Fourier sine transform, we get

$$u(x,z,t) = 1-\frac{4}{\pi^2}\int_0^{\infty}\frac{\sin(\xi x)}{\xi}\int_0^{\infty}\frac{\sin(\varsigma z)}{\varsigma}\times\exp\left(-\frac{\xi^2+\varsigma^2}{1+\eta(\xi^2+\varsigma^2)}t\right)\mathrm{d}\varsigma\,\mathrm{d}\xi. \tag{62}$$

This is just the dimensionless velocity solution for a second grade fluid obtained by Fetecau [4].

4.2. Solution of temperature field

Using the non-dimensional quantities given by Eq. (35) and $z^* = zU/v$, Eqs. (48), (51), (52), and (54) reduce to dimensionless equations as follows (for brevity the dimensionless mark "*" are omitted here)

$$\frac{1}{Pr}\left[\frac{\partial^2\theta(x,z,t)}{\partial x^2}+\frac{\partial^2\theta(x,z,t)}{\partial z^2}\right]+\lambda\left\{\left[\frac{\partial u(x,z,t)}{\partial x}\right]^2+\left[\frac{\partial u(x,z,t)}{\partial z}\right]^2\right\} = \frac{\partial\theta(x,z,t)}{\partial t}, \tag{63}$$

$$\theta(x,z,0)=0 \quad \text{for } x>0,\quad z>0, \tag{64}$$

$$\theta(0,z,t)=\theta(x,0,t)=1 \quad \text{for } t>0, \tag{65}$$

$$\theta(x,z,t),\ \frac{\partial\theta(x,z,t)}{\partial x},\ \frac{\partial\theta(x,z,t)}{\partial z}\to 0 \quad \text{for } x^2+z^2\to\infty. \tag{66}$$

Letting

$$g(x,z,t)=\lambda\left\{\left[\frac{\partial u(x,z,t)}{\partial x}\right]^2+\left[\frac{\partial u(x,z,t)}{\partial z}\right]^2\right\},$$

Eq. (63) can be rewritten as

$$\frac{1}{Pr}\left[\frac{\partial^2\theta(x,z,t)}{\partial x^2}+\frac{\partial^2\theta(x,z,t)}{\partial z^2}\right]+g(x,z,t)=\frac{\partial\theta(x,z,t)}{\partial t}. \tag{67}$$

Using the same method as that in Section 3.2, we get

$$\theta(x,z,t)=\frac{2}{\pi}\int_0^\infty\int_0^\infty \sin(\xi x)\sin(\varsigma z)\mathrm{e}^{-(\xi^2+\varsigma^2)t/Pr}\cdot\int_0^t\left[\frac{2T}{\pi Pr}\frac{\xi^2+\varsigma^2}{\xi\varsigma}+g_s(\xi,\varsigma,\tau)\right] \times \mathrm{e}^{(\xi^2+\varsigma^2)\tau/Pr}\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}\varsigma. \tag{68}$$

5. Conclusions

In this work, we have presented some results about the generalized second fluid on a heated flat plate and within a heated edge. Exact solutions of the velocity and temperature fields are obtained using the Fourier sine transform and fractional Laplace transform. The temperature distribution in a generalized second grade fluid subject to a linear flow on a heated flat plate and within a heated edge was determined. Some previous and classical results can be considered as particular cases of our results, such as the solutions corresponding to a second grade fluids and Newtonian fluids. They can be easily obtained from those by letting $\beta=1$ and $\beta=0$, $\alpha=0$, respectively. It is also shown that the fractional constitutive relationship model is more flexible than the conventional model in describing the properties of viscoelasitic fluid.

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